From Newton to Boltzmann: the case of short-range potentials

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Isabelle Gallagher, Laure Saint-Raymond, Benjamin Texier

Abstract. — We fill in all details in the proof of Lanford's theorem. This provides a rigorous derivation of the Boltzmann equation as the mesoscopic limit of systems of Newtonian particles interacting via a short-range potential, as the number of particles N goes to infinity and the characteristic length of interaction ε simultaneously goes to 0, in the Boltzmann-Grad scaling $N\varepsilon^{d-1} \equiv 1$. The case of localized elastic interactions, i.e., hard spheres, is a corollary of the proof. The time of validity of the convergence is a fraction of the mean free time between two collisions, due to a limitation of the time on which one can prove the existence of the BBGKY and Boltzmann hierarchies. Our proof relies on the important contributions of King, Cercignani, Illner and Pulvirenti, and Cercignani, Gerasimenko and Petrina.

CONTENTS

CHAPTER 1

THE BOLTZMANN-GRAD LIMIT

We study the qualitative behavior of systems of interacting particles of the form

$$\frac{dx_i}{dt} = v_i \,, \qquad m_i \frac{dv_i}{dt} = -\sum_{j \neq i} \nabla\Phi(x_i - x_j) \,, \tag{1.0.1}$$

for $1 \leq i \leq N$, where $(x_i, v_i) \in \mathbf{R}^d \times \mathbf{R}^d$ denote position and velocity of particle i with mass m_i (which we shall assume equal to 1 to simplify) and the force exerted by particle j on particle i is $-\nabla\Phi(x_i - x_j)$.

– When the system is constituted of two elementary particles, in the reference frame attached to the center of mass, the dynamics is one-dimensional. The deflection of the particle trajectories from straight lines can then be described through explicit formulas (which will be given in Chapter 3).

– When the system is constituted of three particles or more, the integrability is lost, and in general the problem becomes very complicated, as already noted by Poincaré [**34**].

1.1. Thermodynamic limit

In the large N limit, called the thermodynamic limit, individual trajectories become irrelevant, and the goal is to describe an average behaviour.

The Liouville equation relative to the particle system (1.0.1) is

$$\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N - \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \nabla_x \Phi\,(x_i - x_j) \cdot \nabla_{v_i} f_N = 0 \,.$$

We use the following notation: for any set of s particles with positions $X_s := (x_1, \ldots, x_s) \in \mathbf{R}^{ds}$ and velocities $V_s := (v_1, \ldots, v_s) \in \mathbf{R}^{ds}$, we write $z_i := (x_i, v_i) \in \mathbf{R}^{2d}$ and $Z_s := (z_1, \ldots, z_s) \in \mathbf{R}^{2ds}$. We assume that the probability f_N, referred to as the N-particle distribution function, satisfies for all permutations σ of $\{1, \ldots, N\}$,

$$f_N(t, Z_{\sigma(N)}) = f_N(t, Z_N) \,, \tag{1.1.1}$$

with $Z_{\sigma(N)} = (x_{\sigma(1)}, v_{\sigma(1)}, \ldots, x_{\sigma(N)}, v_{\sigma(N)})$. This corresponds to the property that the particles are indistinguishable.

The average behavior of the particles is then described by the first marginal $f_N^{(1)}$ of the distribution function f_N, defined by

$$f_N^{(1)}(t, Z_1) := \int f_N(t, Z_N) dz_2 \dots dz_N \,.$$

In this framework, in order for the average energy per particle to remain bounded, one has to assume that the energy of each pairwise interaction is small. In other words, one has to consider a rescaled potential Φ_ε obtained
– either by scaling the strength of the force,
– or by scaling the range of potential.

1.2. Mean field versus collisional dynamics

According to the scaling chosen, we expect to obtain different asymptotics.

• In the case of a weak coupling, i.e. when the strength of the individual interaction becomes small (of order $1/N$) but the range remains macroscopic, the convenient scaling in order for the macroscopic dynamics to be sensitive to the coupling is:

$$\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N - \frac{1}{N} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \nabla_x \Phi\left(x_i - x_j\right) \cdot \nabla_{v_i} f_N = 0 \,.$$

Then each particle feels the effect of the force field created by all the (other) particles

$$F_N(x) = -\frac{1}{N} \sum_{j=1}^N \nabla_x \Phi\left(x - x_j\right) \sim -\iint \nabla_x \Phi(x-y) f_N^{(1)}(t, y, v) dy dv \,.$$

In particular, the dynamics seems to be stable under small perturbations of the positions or velocities of the particles.

In the thermodynamic limit, we thus get a *mean field approximation*, that is an equation of the form

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0$$

for the first marginal, where the coupling arises only through some average

$$F := -\nabla_x \Phi * \int f dv \,.$$

An important amount of literature is devoted to such asymptotics, but this is not our purpose here. We refer to [**10, 38**] for pioneering results, to [**22**] for a recent study and to [**19**] for a review on that topic.

• The scaling we shall deal with in the present work corresponds to a strong coupling, i.e. to the case when the amplitude of the potential remains of size $O(1)$, but its range becomes small. We shall assume throughout this text the following properties for Φ (a *short-range* potential).

Assumption 1.2.1. — *The potential* $\Phi : \mathbf{R}^d \to \mathbf{R}$ *is a radial, nonnegative, nonincreasing function supported in the unit ball of* $\mathbf{R}^d$*, of class* C^2 *in* $\{x \in \mathbf{R}^d\,, 0 < |x| < 1\}$*. Moreover it is assumed that* Φ *is unbounded near zero, goes to zero at* $|x| = 1$ *with bounded derivatives, and that* $\nabla\Phi$ *vanishes only on* $|x| = 1$*. Finally writing* $\Phi(x) = \Phi(|x|)$ *we assume that for all* $\rho \in (0, 1)$*,*

$$\rho\Phi''(\rho) + 2\Phi'(\rho) \geq 0\,. \tag{1.2.1}$$

Remark 1.2.2. — *We refer to Chapter* 3 *for a justification of those assumptions, in particular* (1.2.1) *that appears as a sufficient condition to define a scattering cross-section. Condition* (1.2.1) *can easily be checked for a large class of potentials. For instance any potential of the form* $\Phi(\rho) = \rho^{-k} - 1$ *for* $\rho < 1$ *is suitable (for* $k \geq 1$*). Potentials smooth at* $\rho = 1$ *can be constructed from that example by using a smooth junction* ([**36**]).

Introduce a small parameter $\varepsilon > 0$ corresponding to the typical interaction length of the particles. Then in the macroscopic spatial and temporal scales, the Hamiltonian system becomes

$$\frac{dx_i}{dt} = v_i\,, \qquad \frac{dv_i}{dt} = -\frac{1}{\varepsilon}\sum_{j\neq i}\nabla\Phi\left(\frac{x_i - x_j}{\varepsilon}\right), \tag{1.2.2}$$

and the Liouville equation takes the form

$$\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N - \sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \frac{1}{\varepsilon}\nabla_x\Phi\left(\frac{x_i - x_j}{\varepsilon}\right)\cdot\nabla_{v_i} f_N = 0\,. \tag{1.2.3}$$

With such a scaling, the dynamics is very sensitive to the positions of the particles.

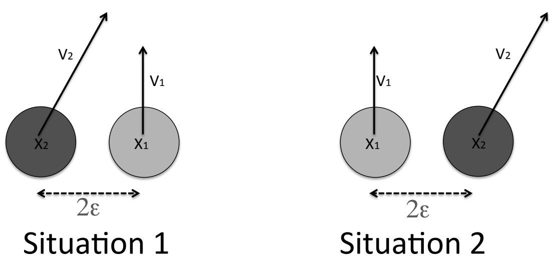

FIGURE 1. Instability

Situations 1 and 2 on Figure 1 are different by a spatial translation of $O(\varepsilon)$ only. However in Situation 1, particles will interact and be deviated from their free motion, while in Situation 2, they will evolve under free flow.

1.3. The Boltzmann equation

Of course, particles move with uniform rectilinear motion as long as they remain at a distance greater than ε to other particles. In the limit $\varepsilon \to 0$, we thus expect trajectories to be almost polylines.

Deflections are due to elementary interactions
- which occur when two particles are at a distance smaller than ε,
- during a time interval of order ε (if the relative velocity is not too small),
- which involve generally only two particles : the probability that a third particle enters a security ball of radius ε should indeed tend to 0 as $\varepsilon \to 0$ in the convenient scaling. We are therefore brought back to the case of the two-body system, which is completely integrable (see Chapter 3).

In order for the interactions to have a macroscopic effect on the dynamics, each particle should undergo a finite number of collisions per unit of time. A scaling argument, giving the mean free path in terms of N and ε, then shows that $N\varepsilon^{d-1} = O(1)$. This is the Boltzmann-Grad scaling (see [**21**]).

In the limit $\varepsilon \to 0$ with $N\varepsilon^{d-1} = 1$, we would like to obtain a kind of homogeneisation result : we want to average the motion over the small scales in t and x, and replace the localized interactions by pointwise collisions as in the case of hard spheres.

We shall therefore introduce an artificial boundary (following [**27**]) so that
- on the exterior domain, the dynamics reduces to free transport,
- on the interior domain, the dynamics can be integrated in order to compute outwards boundary conditions in terms of the incoming flux. Note that such a scattering operator is relevant only if we can guarantee that there is no other particle involved in the interaction.

The statistical distribution of deflection angles ω is then predicted by a function $b = b(v - v_1, \omega)$, the collision cross-section, which depends only on the microscopic interaction potential.

A counting argument leads then to the Boltzmann equation (introduced by Boltzmann in [**7**]-[**8**]) :

$$\left\{ \begin{array}{rcl} \underbrace{\partial_t f + v \cdot \nabla_x f}_{\text{free transport}} & = & \underbrace{Q(f,f)}_{\text{localized binary collisions}} \\ Q(f,f) & := & \displaystyle\iint [f(v')f(v_1') - f(v)f(v_1)] b(v - v_1, \omega) dv_1 d\omega \,. \\ & & \qquad v',\, v_1' \text{ pre-collisional velocities} \end{array} \right. \tag{1.3.1}$$

The collision term, which acts only on the v-variable, is constituted of a gain term, corresponding to the creation of particles of velocity v by collision between particles of velocities v' and v_1', and of a loss term, due to the disappearance of particles of velocity v by collision with particles of velocity v_1.

Note that the joint probability of having particles of velocity (v', v_1') (respectively of velocities (v, v_1)) before the collision is assumed to be equal to $f(t, x, v')f(t, x, v_1')$ (resp. to $f(t, x, v)f(t, x, v_1)$), meaning that there is independence.

1.4. A convergence result

The goal of this text is to prove the following statement. It is written somewhat loosely, we refer to Chapter 7 for a precise statement (see in particular the statement of Theorem 4 page 53). The appropriate notion of independence is defined in Section 7.1 and the appropriate notion of convergence is defined in Section 7.2.

Theorem 1. — *Assume that the repulsive potential* Φ *satisfies Assumption* 1.2.1. *Let* $f_0 : \mathbf{R}^{2d} \mapsto \mathbf{R}^+$ *be a continuous function, of integral one, and with exponential decay at large energies. Consider the system of* N *particles, initially distributed according to* f_0 *and "independent" (in a sense made precise in Chapter* 7*), governed by the system* (1.2.2)*. Then, in the Boltzmann-Grad limit* $N \to \infty$, $N\varepsilon^{d-1} \sim 1$, *its distribution function converges to the solution to the Boltzmann equation* (1.3.1) *with initial data* f_0*, in the sense of observables.*

1.5. Related results

The problem of asking for a rigorous derivation of the Boltzmann equation from the Hamiltonian dynamics goes back to Hilbert [**24**], who suggested to use the Boltzmann equation as an intermediate step between the Hamiltonian dynamics and fluid mechanics, and who described this axiomatization of physics as a major challenge for mathematicians of the twentieth century.

We shall not give an exhaustive presentation of the studies that have been carried out on this question but indicate some of the fundamental landmarks. First one should mention N. Bogoliubov [**5**], M. Born, and H. S. Green [**9**], J. G. Kirkwood [**28**] and J. Yvan [**42**], who gave their names to the BBGKY hierarchy we shall be using extensively in this study. H. Grad was able to obtain in [**20**] a differential equation on the first marginal which after some manipulations converges towards the Boltzmann equation. The first mathematical result on this problem goes back to C. Cercignani [**11**] and O. Lanford [**31**] who proved the propagation of chaos by a careful study of trajectories of a hard spheres system, and who exhibited – for the first time – the origin of irreversibility. The proof, even though incomplete, is therefore an important breakthrough. The limits of their methods, on which we will comment later on – especially regarding the short time of convergence – are still challenging questions.

The argument of O. Lanford was then revisited and completed in several works. Let us mention especially the contributions of K. Uchiyama [**39**], C. Cercignani, R. Illner and M. Pulvirenti [**14**] and H. Spohn [**37**] who introduced a mathematical formalism, in particular for the existence of solutions to the BBGKY hierarchy which turns out to be a theory in the spirit of the Cauchy-Kowalewskaya theorem.

The term-by-term convergence of the hierarchy in the Boltzmann-Grad scaling was studied in more details by Cercignani, V. I. Gerasimenko and D. I. Petrina [**13**] : it indeed requires refined estimates on the set of "pathological trajectories", i.e. trajectories for which the Boltzmann equation does not provide a good approximation of the dynamics.

The method of proof was extended
– to the case when the initial distribution is close to vacuum, in which case global in time results may be proved [**14, 25, 26**];
– to the case when interactions are localized but not pointwise [**27**]. Because multiple collisions are no longer negligible, this requires a careful study of clusters of particles.

Many review papers deal with those different results, see [**17, 35, 41**] for instance. Our goal here is to provide an elementary and self-contained presentation, which includes all the details of the proofs, especially concerning convergence which to our knowledge is not completely written anywhere, and concerning reasonable assumptions that can be made on the potential.

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CHAPTER 2

FORMAL DERIVATION OF THE BOLTZMANN EQUATION FOR HARD-SPHERES

This chapter is intended to familiarize the reader with the methods and notation related to BBGKY and Boltzmann hierarchies. We present formally, in the situation of hard-spheres, the passage from the Liouville equation associated with the N-particle flow to the BBGKY hierarchy, and then the limit to the Boltzmann equation. All the results stated here will be proved – in the more complicated case of nonlocal interactions via a potential – in the next chapters. Readers already familiar with the subject may skip this chapter altogether.

2.1. The N-particle flow

We consider N particles, the motion of which is described by N positions $(x_1, \dots, x_N)$ and N velocities $(v_1, \dots, v_N)$, each in $\mathbf{R}^d$. Denoting by $Z_N := (z_1, \dots, z_N)$ the set of particles, each particle $z_i := (x_i, v_i) \in \mathbf{R}^{2d}$ is submitted to free flow on the domain

$$\mathcal{D}_N := \Big\{ Z_N \in \mathbf{R}^{2dN} \,/\, \forall i \neq j,\, |x_i - x_j| > \varepsilon \Big\}$$

and bounces off the boundary $\partial\mathcal{D}_N$ according to the laws of elastic reflection:

$$\begin{aligned}
&\frac{dx_i}{dt} = v_i \,, \quad \frac{dv_i}{dt} = 0 \;\text{ on }\; \mathcal{D}_N \\
&v_i^{in} = v_i^{out} - (v_i^{out} - v_j^{out}) \cdot \nu^{i,j}\, \nu^{i,j} \\
&v_j^{in} = v_j^{out} + (v_i^{out} - v_j^{out}) \cdot \nu^{i,j}\, \nu^{i,j} \;\text{ if }\; \exists j \neq i \,, |x_i - x_j| = \varepsilon \,,
\end{aligned} \tag{2.1.1}$$

where $\nu^{i,j} := (x_i - x_j)/|x_i - x_j|$, and in the case when $\nu^{i,j} \cdot (v_i^{in} - v_j^{in}) < 0$ (meaning that the ingoing velocities are precollisional).

Contrary to the potential case studied in the next chapters, it is not obvious to check that (2.1.1) defines a global dynamics, at least for almost all initial data. Note that this is not a simple consequence of the Cauchy-Lipschitz theorem since the boundary condition is not smooth, and even not defined for all configurations. In the presence of a potential, we shall prove in this text that the set of trajectories involving multiple collisions has zero measure. Let us prove this result for the hard sphere dynamics: we call pathological a trajectory such that

- either there exists a collision involving more than two particles, hence the boundary condition is not well defined;

- or there are an infinite number of collisions in finite time so the dynamics cannot be globally defined.

In [**2**], Proposition 4.3 it is stated that outside a negligible set of initial data there are no pathological trajectories; the complete proof is provided in [**1**]. Actually the setting of [**1**] is more complicated than ours since the case of an infinite number of particles is considered. The arguments of [**1**] can however be easily adapted to our case to yield the following result, whose proof we detail for the convenience of the reader.

Proposition 2.1.1. — *Let N, ε be fixed. The set of initial configurations leading to a pathological trajectory is of measure zero in $\mathbf{R}^{2dN}$.*

We first prove the following elementary lemma.

Lemma 2.1.2. — *Let $\rho, R > 0$ be given, and $\delta < \varepsilon/2$. Define*

$$I := \Big\{ Z_N \in B_\rho^N \times B_R^N \, / \, \textit{one particle will collide with two others on the time interval } [0,\delta] \Big\} .$$

Then $|I| \leq C(N,\varepsilon,R)\, \rho^{d(N-2)} \delta^2$.

Proof. — Just notice that I is embeded in

$$\Big\{ Z_N \in B_\rho^N \times B_R^N \, / \, \exists \{i,j,k\} \text{ distinct}\, , \quad |x_i - x_j| \in [\varepsilon, \varepsilon + 2R\delta] \quad \text{and} \quad |x_i - x_k| \in [\varepsilon, \varepsilon + 2R\delta] \Big\} ,$$

and the lemma follows directly. $\square$

Proof of Proposition 2.1.1. — Now let $R > 0$ be given and fix some time $t > 0$. Let $\delta < \varepsilon/2$ be a parameter such that t/δ is an integer.

Lemma 2.1.2 implies that there is a subset $I_0(\delta, R)$ of $B_R^N \times B_R^N$ of measure at most $C(N,\varepsilon,R)R^{d(N-2)}\delta^2$ such that any initial configuration belonging to $(B_R^N \times B_R^N) \setminus I_0(\delta,R)$ generates a solution on $[0,\delta]$ such that each particle encounters at most one other particle on $[0,\delta]$.

Now let us start again at time δ. We recall that in the velocity variables, the ball of radius R in $\mathbf{R}^{dN}$ is stable by the flow, whereas the positions at time δ lie in the ball $B_{R+R\delta}^N$. Let us apply Lemma 2.1.2 again to that new initial configuration space. Since the measure is invariant by the flow, we can construct a subset $I_1(\delta,R)$ of the initial positions $B_R^N \times B_R^N$, of size $C(N,\varepsilon,R)R^{d(N-2)}(1+\delta)^{d(N-2)}\delta^2$ such that outside $I_0 \cup I_1(\delta,R)$, the flow starting from any initial point in $B_R^N \times B_R^N$ is such that each particle encounters at most one other particle on $[0,\delta]$, and then at most one other particle on $[\delta, 2\delta]$.

We repeat the procedure t/δ times: we construct a subset $I_\delta(t,R) := \displaystyle\bigcup_{j=0}^{t/\delta-1} I_j(\delta,R)$ of $B_R^N \times B_R^N$, of measure

$$|I_\delta(t,R)| \leq C(N,\varepsilon,R)R^{d(N-2)}\delta^2 \sum_{j=0}^{t/\delta-1} (1+j\delta)^{d(N-2)} \leq C(N,R,t,\varepsilon)\delta$$

such that for any initial configuration in $B_R^N \times B_R^N$ outside that set, the flow is well-defined up to time t. The intersection $I(t,R) := \displaystyle\bigcap_{\delta>0} I_\delta(t,R)$ is of measure zero, and any initial configuration in $B_R^N \times B_R^N$ outside $I(t,R)$ generates a well-defined flow until time t. Finally we consider the countable union of those zero measure sets $I := \displaystyle\bigcup_n I(t_n,R_n)$ where t_n and R_n go to infinity, and any initial configuration in $\mathbf{R}^{2dN}$ outside I generates a globally defined flow. The proposition is proved. $\square$

2.2. The Liouville equation and the BBGKY hierarchy

The Liouville equation relative to the particle system (2.1.1) is

$$\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N = 0 \quad \text{in} \quad \mathcal{D}_N \tag{2.2.1}$$

with the boundary condition $f_N(t, Z_N^{in}) = f_N(t, Z_N^{out})$. We assume from now on that f_N is invariant by permutation in the sense of (1.1.1), meaning that the particles are indistinguishable.

One can associate with this Liouville equation a hierarchy of equations, satisfied by the marginals

$$f_N^{(s)}(t, Z_s) := \int_{\mathbf{R}^{2d(N-s)}} f_N(t, Z_s, z_{s+1}, \ldots, z_N) \mathbb{1}_{Z_N \in \mathcal{D}_N} \, dz_{s+1} \ldots dz_N \, .$$

Let us derive this hierarchy formally. We integrate (2.2.1) over the $(N-s)$ last variables, and we first notice that

$$\int_{\mathbf{R}^{2d(N-s)}} \partial_t f_N(t, Z_s, z_{s+1}, \ldots, z_N) \mathbb{1}_{Z_N \in \mathcal{D}_N} \, dz_{s+1} \ldots dz_N = \partial_t f_N^{(s)}(t, Z_s) \, .$$

Next we compute

$$\sum_{i=1}^N \int_{\mathbf{R}^{2d(N-s)}} v_i \cdot \nabla_{x_i} f_N(t, Z_N) \mathbb{1}_{Z_N \in \mathcal{D}_N} \, dz_{s+1} \ldots dz_N$$

using Green's formula. The boundary terms involve configurations with at least one pair (i,j), satisfying $1 \leq i \leq N$ and $s+1 \leq j \leq N$, with $|x_i - x_j| = \varepsilon$. According to the previous section we may neglect configurations where more than two particles collide at the same time, so the boundary condition is well defined. For any $i \in \{1, \ldots, N\}$ and any $j \in \{s+1, \ldots, N\}$, we denote by $\nu(i,j)$ the outwards normal at any point of the boundary and we define

$$\Sigma_N(i,j) := \Big\{ Z_N \in \mathbf{R}^{2dN}, \, |x_i - x_j| = \varepsilon \Big\} \, .$$

Recalling that $\nu^{i,j} := \dfrac{x_i - x_j}{|x_i - x_j|}$ we then obtain, using the invariance of f_N by permutation,

$$\begin{aligned} &\sum_{i=1}^N \int_{\mathbf{R}^{2d(N-s)}} v_i \cdot \nabla_{x_i} f_N(t, Z_N) \mathbb{1}_{Z_N \in \mathcal{D}_N} \, dz_{s+1} \ldots dz_N \\ &\qquad = - \sum_{i=1}^s v_i \cdot \nabla_{x_i} f_N^{(s)}(t, Z_s) + \frac{1}{\sqrt{2}} \sum_{i=1}^s \sum_{j=s+1}^N \int_{\Sigma_N(i,j)} \nu^{i,j} \cdot (v_j - v_i) \, f_N(t, Z_N) \, d\sigma_N^{i,j} \, , \end{aligned}$$

with $d\sigma_N^{i,j}$ the surface measure on $\Sigma_N(i,j)$, induced by the Lebesgue measure.

By symmetry this gives

$$\begin{aligned} &\sum_{i=1}^N \int_{\mathbf{R}^{2d(N-s)}} v_i \cdot \nabla_{x_i} f_N(t, Z_N) \mathbb{1}_{Z_N \in \mathcal{D}_N} \, dz_{s+1} \ldots dz_N \\ &\qquad = - \sum_{i=1}^s v_i \cdot \nabla_{x_i} f_N^{(s)}(t, Z_s) + \frac{N-s}{\sqrt{2}} \sum_{i=1}^s \int_{\Sigma_N(i,s+1)} \nu^{i,s+1} \cdot (v_{s+1} - v_i) \, f_N(t, Z_N) \, d\sigma_N^{i,s+1} \, . \end{aligned}$$

It remains to define the *collision operator*

$$\mathcal{C}_{s,s+1} f_N^{(s+1)}(t, Z_{s+1}) := (N-s) \sum_{i=1}^s \int_{S_\varepsilon(x_i) \times \mathbf{R}^d} \nu^{i,s+1} \cdot (v_{s+1} - v_i) \, f_N^{(s+1)}(t, Z_s, z_{s+1}) \, d\sigma dv_{s+1} \tag{2.2.2}$$

where $S_\varepsilon(x_i)$ is the sphere of radius ε centered at x_i and $d\sigma$ is the surface measure on that sphere and in the end we obtain the BBGKY hierarchy

$$\partial_t f_N^{(s)} + \sum_{1\leq i\leq s} v_i\cdot\nabla_{x_i} f_N^{(s)} = \mathcal{C}_{s,s+1} f_N^{(s+1)} \quad \text{in} \quad \mathcal{D}_s\,, \tag{2.2.3}$$

with the boundary conditions (2.1.1).

2.3. The Boltzmann hierarchy and the Boltzmann equation

Starting from (2.2.3) we now consider the limit $N\to\infty$ under the Boltzmann-Grad scaling $N\varepsilon^{d-1}\equiv 1$. The Duhamel formulation for (2.2.3) writes

$$f_N^{(s)}(t) = \mathbf{D}_s(t) f_{N,0}^{(s)} + \int_0^t \mathbf{D}_s(t-\tau)\mathcal{C}_{s,s+1} f_N^{(s+1)}(\tau)\,d\tau\,,$$

where $\mathbf{D}_s(t)$ denotes the s-particle flow on $\mathcal{D}_s$ with the boundary conditions (2.1.1).

Because of the scaling assumption, the collision term $\mathcal{C}_{s,s+1}f^{(s+1)}(Z_s)$ is approximately equal to

$$-(N-s)\varepsilon^{d-1}\sum_{i=1}^s \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} \nu\cdot(v_{s+1}-v_i) f_N^{(s+1)}(Z_s,x_i+\varepsilon\nu,v_{s+1})\,d\nu dv_{s+1}$$

which we may split into two terms, depending on the sign of $\nu\cdot(v_{s+1}-v_i)$:

$$\begin{aligned}&\sum_{i=1}^s \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} \Big(\nu\cdot(v_{s+1}-v_i)\Big)_+ f_N^{(s+1)}(Z_s,x_i+\varepsilon\nu,v_{s+1})\,d\nu dv_{s+1}\\ &\quad -\sum_{i=1}^s \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} \Big(\nu\cdot(v_{s+1}-v_i)\Big)_- f_N^{(s+1)}(Z_s,x_i+\varepsilon\nu,v_{s+1})\,d\nu dv_{s+1}\,.\end{aligned}$$

Recall that pre-collisional particles are particles (x_i,v_i) and (x_{s+1},v_{s+1}) for which

$$(x_{s+1}-x_i)\cdot(v_{s+1}-v_i)<0\,.$$

The case when $(x_{s+1}-x_i)\cdot(v_{s+1}-v_i)>0$ is called the post-collisional case. Consider a set of particles Z_{s+1} such that (x_i,v_i) and (x_{s+1},v_{s+1}) are post-collisional. Provided that there is no other particle among these $s+1$ which has undergone a collision on a short time interval (which is almost sure in the limit $\varepsilon\to 0$), we have

$$f_N^{(s+1)}(t,Z_s,x_{s+1},v_{s+1}) = f_N^{(s+1)}(t,Z_s^*,x_{s+1}^*,v_{s+1}^*)$$

where (z_i^*,z_{s+1}^*) is the pre-image of (z_i,z_{s+1}) by (2.1.1). Then neglecting the small spatial translations in the arguments of $f_N^{(s+1)}$, we obtain the following asymptotic expression for the collision operator at the limit:

$$\begin{aligned}\mathcal{C}^0_{s,s+1} f^{(s+1)}(t,Z_s) &:= \sum_{i=1}^s \int \mathbb{1}_{\nu\cdot(v_{s+1}-v_i)>0}\,\nu\cdot(v_{s+1}-v_i)\\ &\times\Big(f^{(s+1)}(t,x_1,v_1,\dots,x_i,v_i^*,\dots,x_s,v_s,x_i,v_{s+1}^*) - f^{(s+1)}(t,Z_s,x_i,v_{s+1})\Big)d\nu dv_{s+1}\,.\end{aligned}$$

The asymptotic dynamics are therefore governed by the following Boltzmann hierarchy:

$$f^{(s)}(t) = \mathbf{S}_s(t) f_0^{(s)} + \int_0^t \mathbf{S}_s(t-\tau)\mathcal{C}^0_{s,s+1} f^{(s+1)}(\tau)\,d\tau\,. \tag{2.3.1}$$

where $\mathbf{S}_s(t)$ denotes free-flow.

Note that if $f^{(s)}(t, Z_s) = \prod_{i=1}^{s} f(t, z_i)$ (meaning $f^{(s)}(t)$ is *tensorized*) then f satisfies the Boltzmann equation (1.3.1), where the cross-section b is simply $b(v_{s+1} - v_i, \omega) := \mathbb{1}_{\omega\cdot(v_{s+1}-v_i)>0}\, \omega \cdot (v_{s+1} - v_i)$.

CHAPTER 3

TWO-PARTICLE INTERACTIONS

In the case when the microscopic interaction between particles is governed by a short-range repulsive potential, collisions are no more instantaneous and pointwise, and they possibly involve more than two particles. Our analysis in Chapters 8 to 10 shows however that the low density limit $N\varepsilon^{d-1} \to 0$ requires only a description of two-particle interactions, at the exclusion of more complicated interactions.

In this chapter we therefore study precisely, following the lines of [**12**], the Hamiltonian system (1.2.2) for $N = 2$. The study of the reduced motion is carried out in Section 3.1, while the scattering map is introduced in Section 3.2, and the cross-section, which will play in important role in the Boltzmann hiearchy, is described in Section 3.3.

3.1. Reduced motion

We first define a notion of pre- and post-collisional particles, by analogy with the dynamics of hard spheres:

Definition 3.1.1. — *Two particles z_1, z_2 are said to be pre-collisional if they belong to the artificial boundary and their distance is decreasing:*

$$|x_1 - x_2| = \varepsilon, \qquad (v_1 - v_2) \cdot (x_1 - x_2) < 0.$$

Two particles z_1, z_2 are said to be post-collisional if they belong to the artificial boundary and their distance is increasing:

$$|x_1 - x_2| = \varepsilon, \qquad (v_1 - v_2) \cdot (x_1 - x_2) > 0.$$

We consider here only two-particle systems, and show in Lemma 3.1.2 that, if z_1 and z_2 are pre-collisional at time t_-, then there exists a post-collisional configuration z_1', z_2', attained at $t_+ > t_-$. Since $\nabla\Phi(x)$ vanishes on $\{|x| \geq \varepsilon\}$, the particles z_1 and z_2 travel at constant velocities v_1' and v_2' for ulterior ($t > t_+$) times.

Momentarily changing back the macroscopic scales of (1.2.2) to the microscopic scales of (1.0.1) by defining $\tau := (t - t_-)/\varepsilon$ and $y(\tau) := x/\varepsilon(\tau)$, $w(\tau) = v(\tau)$, we find that the two-particle dynamics is governed by the equations

$$\left\{\begin{aligned} &\frac{dy_1}{d\tau} = w_1\,, \quad \frac{dy_2}{d\tau} = w_2\,,\\ &\frac{dw_1}{d\tau} = -\nabla\Phi\,(y_1-y_2) = -\frac{dw_2}{d\tau}\,, \end{aligned}\right. \tag{3.1.1}$$

whence the conservations

$$\frac{d}{d\tau}(w_1+w_2)=0\,, \qquad \frac{d}{d\tau}\left(\frac14(w_1+w_2)^2+\frac14(w_1-w_2)^2+\Phi(y_1-y_2)\right)=0\,. \tag{3.1.2}$$

From (3.1.2) we also deduce that the center of mass has a uniform, rectilinear motion:

$$(y_1+y_2)(\tau)=(y_1+y_2)(0)+\tau(w_1+w_2)\,, \tag{3.1.3}$$

and that pre- and post-collisional velocities are linked by the classical relations

$$w_1'+w_2'=w_1+w_2, \quad |w_1'|^2+|w_2'|^2=|w_1|^2+|w_2|^2\,. \tag{3.1.4}$$

A consequence of (3.1.1) is that $(\delta y,\delta w):=(y_1-y_2,w_1-w_2)$ solves

$$\frac{d}{d\tau}\delta y=\delta w\,, \qquad \frac{d}{d\tau}\delta w=-2\nabla\Phi(\delta y)\,. \tag{3.1.5}$$

We notice that, Φ being radial, there holds

$$\frac{d}{d\tau}(\delta y\wedge\delta w)=\delta w\wedge\delta w-2\delta y\wedge\nabla\Phi(\delta y)=0\,,$$

implying that, if the initial angular momentum $\delta y_0\wedge\delta w_0$ is non-zero, then δy remains for all times in the hyperplane orthogonal to $\delta y_0\wedge\delta w_0$. In this hyperplane, introducing spherical coordinates (ρ,φ) in $\mathbf{R}_+\times\mathbf{S}_1^{d-2}$, such that

$$\delta y=\rho e_\rho \quad \text{and} \quad \delta w=\dot\rho e_\rho+\rho\dot\varphi e_\varphi$$

the conservations of energy and angular momentum take the form

$$\begin{aligned} \frac12(\dot\rho^2+(\rho\dot\varphi)^2)+2\Phi(\rho)&=\frac12|\delta w_0|^2\,,\\ \rho^2|\dot\varphi|&=|\delta y_0\wedge\delta w_0|\,, \end{aligned}$$

implying $\rho>0$ for all times, and

$$\dot\rho^2+\Psi(\rho,\mathcal{E}_0,\mathcal{J}_0)=\mathcal{E}_0\,, \qquad \Psi:=\frac{\mathcal{E}_0\mathcal{J}_0^2}{\rho^2}+4\Phi(\rho)\,, \tag{3.1.6}$$

where we have defined

$$\mathcal{E}_0:=|\delta w_0|^2 \quad \text{and} \quad \mathcal{J}_0:=|\delta y_0\wedge\delta w_0|/|\delta w_0|=\sin\alpha\,, \tag{3.1.7}$$

which are respectively (twice) the energy and the impact parameter, $\pi-\alpha$ being the angle between δw_0 and δy_0 (notice that $\alpha\geq\pi/2$ for pre-collisional situations). In the limit case when $\alpha=0$, the movement is confined to a line since $\dot\varphi\equiv0$.

We consider the sets corresponding to pre- and post-collisional configurations:

$$\mathcal{S}^\pm:=\big\{(\delta y,\delta w)\in\mathbf{S}^{d-1}\times\mathbf{R}^d, \quad \pm\delta y\cdot\delta w>0\big\}\,, \tag{3.1.8}$$

where $\mathbf{S}^{d-1}$ is the unit sphere centered at the origin in $\mathbf{R}^d$; in spherical coordinates pre-collisional configurations correspond to $\rho=1$ and $\dot\rho<0$ while post-collisional configurations are correspond to $\rho=1$ and $\dot\rho>0$.

***Lemma 3.1.2* (Description of the reduced motion).** — *For the differential equation* (3.1.5) *with pre-collisional datum* $(\delta y_0, \delta w_0) \in \mathcal{S}^-$, *there holds* $|\delta y(\tau)| \geq \rho_*$ *for all* $\tau \geq 0$, *with the notation*

$$\rho_* = \rho_*(\mathcal{E}_0, \mathcal{J}_0) := \max\left\{\rho \in (0,1), \quad \Psi(\rho, \mathcal{E}_0, \mathcal{J}_0) = \mathcal{E}_0\right\}, \tag{3.1.9}$$

and for τ_* *defined by*

$$\tau_* := 2 \int_{\rho_*}^{1} \left(\mathcal{E}_0 - \Psi(\rho, \mathcal{E}_0, \mathcal{J}_0)\right)^{-1/2} d\rho\,, \tag{3.1.10}$$

the configuration is post-collisional ($\rho = 1$, $\dot\rho > 0$) *at* $\tau = \tau_*$.

Proof. — Solutions to (3.1.6) satisfy $\dot\rho = \iota(\rho)\big(\mathcal{E}_0 - \Psi(\rho)\big)^{1/2}$, with $\iota(\rho) = \pm 1$, possibly changing values only on $\{\Psi = \mathcal{E}_0\}$, by Darboux's theorem (a derivative function satisfies the intermediate value theorem). The initial configuration being pre-collisional, there holds initially $\iota = -1$, corresponding to a decreasing radius. The existence of ρ_* satisfying (3.1.9) is then easily checked: we have $|\delta y_0| = 1$ and $\delta y_0 \cdot \delta w_0 \neq 0$, so there holds $\Psi(1, \mathcal{E}_0, \mathcal{J}_0) < \mathcal{E}_0$, and Ψ is increasing as ρ is decreasing. The set $\{\tau \geq 0, \rho(\tau) \geq \rho_*\}$ is closed by continuity. It is also open: since Φ is nonincreasing, then $\partial_\rho \Psi \neq 0$ everywhere and in particular at $(\rho_*, \mathcal{E}_0, \mathcal{J}_0)$. So $\mathcal{E}_0 - \Psi$ changes sign at ρ_*, which forces, by (3.1.6), the sign function ι to jump from $-$ to $+$ as ρ reaches the value ρ_* from above. This proves $\rho \geq \rho_*$ by connexity. The minimal radius $\rho = \rho_*$ is attained at $\tau_*/2$, where τ_* is defined by (3.1.10), the integral being finite since $\partial_\rho \Psi$ does not vanish. Assume finally that for all $\tau > 0$, there holds $\rho(\tau) < 1$. Then on $[\tau_*/2, +\infty)$, ρ is increasing and bounded, hence converges to a limit radius, which contradicts the definition of ρ_*. This proves $\rho = 1$ at $\tau = \tau_*$, a time at which $\dot\rho > 0$, since ι has jumped exactly once, by definition of ρ_*. □

Remark 3.1.3. — *Denoting* $A : (y, w) \to (y, -w)$, *and* $\phi_t : \mathbf{R}^{2d} \to \mathbf{R}^{2d}$ *the flow of* (3.1.5), *we find that* $\phi_{-t} = A \circ \phi_t \circ A$, *implying* $A \circ \phi_t \circ A \circ \phi_t \equiv \mathrm{Id}$, *and time-reversibility of the two-particle dynamics.*

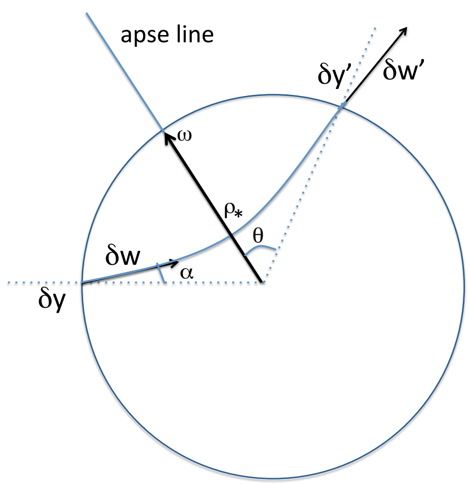

FIGURE 2. Reduced dynamics

The reduced dynamics is pictured on Figure 2, where the half-deflection angle θ is the integral of the angle φ as a function of ρ over $[\rho_*, 1]$:

$$\theta = \int_{\rho_*}^{1} \frac{\mathcal{E}_0^{1/2}\mathcal{I}_0}{\rho^2} \left(\mathcal{E}_0 - \Psi(\rho, \mathcal{E}_0, \mathcal{I}_0)\right)^{-1/2} d\rho\,, \tag{3.1.11}$$

With the initialization choice $\varphi_0 = 0$, the post-collisional configuration is $(\rho, \varphi)(\tau_*) = (1, 2\theta)$; it can be deduced from the pre-collisional configuration by symmetry with respect to the apse line, which by definition is the line through the origin and the point of closest approach $(\delta y(\tau_*/2), \delta w(\tau_*/2))$. The direction of this line is denoted $\omega \in \mathbf{S}^{d-1}$.

3.2. Scattering map

We shall now define a microscopic scattering map $\tilde\sigma_0$ that sends pre- to post-collisional configurations:

$$\tilde\sigma_0 : \quad (\delta y_0, \delta w_0) \in \mathcal{S}^- \longrightarrow (\delta y(\tau_*), \delta w(\tau_*)) = \phi_{\tau_*}(\delta y_0, \delta w_0) \in \mathcal{S}^+\,.$$

By uniqueness of the trajectory of (3.1.5) issued from $(\delta y_0, \delta w_0)$ (a consequence of the regularity assumption on the potential, via the Cauchy-Lipschitz theorem), the scattering is one-to-one. It is also onto, by Remark 3.1.3: the pre-image of $(\delta y, \delta w) \in \mathcal{S}^+$ by the scattering is $I \circ \phi_{\tau_*}(\delta y, -\delta w) \in \mathcal{S}^-$.

Back in the macroscopic variables, we now define a corresponding scattering operator for the two-particle dynamics. In this view, we introduce the sets

$$\mathcal{S}_\varepsilon^\pm := \Big\{(z_1, z_2) \in \mathbf{R}^{4d},\ |x_1 - x_2| = \varepsilon,\ \pm(x_1 - x_2)\cdot(v_1 - v_2) > 0\Big\}\,.$$

We define, as in (3.1.7),

$$\mathcal{E}_0 = |v_1 - v_2|^2 \quad \text{and} \quad \mathcal{J}_0 := \frac{|(x_1 - x_2)\wedge(v_1 - v_2)|}{\varepsilon|v_1 - v_2|} =: \sin\alpha. \tag{3.2.1}$$

Definition 3.2.1 **(Scattering operator)**. — *The* scattering operator *is defined as*

$$\sigma_\varepsilon : (x_1, v_1, x_2, v_2) \in \mathcal{S}_\varepsilon^- \longrightarrow (x_1', v_1', x_2', v_2') \in \mathcal{S}_\varepsilon^+\,,$$

where

$$\begin{aligned}
x_1' &:= \frac{1}{2}(x_1 + x_2) + \frac{\varepsilon\tau_*}{2}(v_1 + v_2) + \frac{\varepsilon}{2}\delta y(\tau_*) = -x_1 + \omega\cdot(x_1 - x_2)\omega + \frac{\varepsilon\tau_*}{2}(v_1 + v_2)\,,\\
x_2' &:= \frac{1}{2}(x_1 + x_2) + \frac{\varepsilon\tau_*}{2}(v_1 + v_2) - \frac{\varepsilon}{2}\delta y(\tau_*) = -x_2 - \omega\cdot(x_1 - x_2)\omega + \frac{\varepsilon\tau_*}{2}(v_1 + v_2)\,,\\
v_1' &:= \frac{1}{2}(v_1 + v_2) + \frac{1}{2}\delta w(\tau_*) = v_1 - \omega\cdot(v_1 - v_2)\,\omega\,,\\
v_2' &:= \frac{1}{2}(v_1 + v_2) - \frac{1}{2}\delta w(\tau_*) = v_2 + \omega\cdot(v_1 - v_2)\,\omega\,,
\end{aligned} \tag{3.2.2}$$

where τ_* is the microscopic interaction time, as defined in Lemma 3.1.2, *$(\delta y(\tau_*), \delta w(\tau_*))$ is the microscopic post-collisional configuration: $(\delta y(\tau_*), \delta w(\tau_*)) = \tilde\sigma_0((x_1 - x_2)/\varepsilon, v_1 - v_2)$, and ω is the direction of the apse line. Denoting by $\nu := (x_1 - x_2)/|x_1 - x_2|$ we also define*

$$\sigma_0(\nu, v_1, v_2) := (\nu', v_1', v_2')\,.$$

The above description of (x'_1, v'_1) and (x'_2, v'_2) in terms of ω is deduced from the identities

$$\delta v(\tau_*) = \delta v_0 - 2\omega \cdot \delta v_0\, \omega \quad \text{and} \quad \delta y(\tau_*) = -\delta y_0 + 2\omega \cdot \delta y_0\, \omega$$

in the reduced microscopic coordinates.

By $\partial_\rho \Psi \neq 0$ in $(0,1)$ and the implicit function theorem, the map $(\mathcal{E}, \mathcal{J}) \to \rho_*(\mathcal{E}, \mathcal{J})$ is C^2 just like Ψ. Similarly, $\tau_* \in C^2$. By Definition 3.2.1 and C^1 regularity of $\nabla\Phi$ (Assumption 1.2.1), this implies that the scattering operator σ_ε is C^1, just like the flow map of the two-particle scattering (denoted ϕ in Remark 3.1.3). The scattering σ_ε is also bijective, for the same reason that the microsopic scattering $\tilde\sigma_0$ is bijective; the inverse map is $\sigma_\varepsilon^{-1} := I \circ \sigma_\varepsilon \circ I$, with notation introduced in Remark 3.1.3.

For any $s \in \mathbf{N}^*$, $R > 0$, we denote $B_R^s := \{V_s \in \mathbf{R}^{ds},\ |V_s| \leq R\}$ where $|\cdot|$ is the euclidean norm; we often write $B_R := B_R^1$.

Proposition 3.2.1. — *Let $R > 0$ be given and consider*

$$\mathcal{S}^{\pm}_{\varepsilon,R} := \Big\{ (z_1, z_2) \in (\mathbf{R}^d \times B_R)^2,\ |x_1 - x_2| = \varepsilon,\ \pm (v_1 - v_2) \cdot (x_1 - x_2) > 0 \Big\}\,.$$

The scattering operator σ_ε is a bijection from $\mathcal{S}^-_{\varepsilon,R}$ to $\mathcal{S}^+_{\varepsilon,R}$.

The macroscopic time of interaction $\mathcal{T}_* = \mathcal{T}_*(\mathcal{E}_0, \mathcal{J}_0) := \varepsilon\tau_*$, where τ_* is defined in (3.1.10)*, is uniformly bounded on compact sets of $\mathbf{R}^+ \setminus \{0\} \times [0,1]$.*

Proof. — We already know that σ_ε is a bijection from $\mathcal{S}^-_\varepsilon$ to $\mathcal{S}^+_\varepsilon$. By (3.1.4), it also preserves the velocity bound. Hence σ_ε is bijective $\mathcal{S}^-_{\varepsilon,R} \to \mathcal{S}^+_{\varepsilon,R}$.

Now given $\mathcal{E}_0 > 0$ and $\mathcal{J}_0 \in [0,1]$, we shall show that τ_* can be bounded by a constant depending only on $\mathcal{E}_0$. Since $\Phi(\rho_*) \leq \mathcal{E}_0/4$, then $\rho_* \geq \Phi^{-1}(\mathcal{E}_0/4)$. Let us then define $i_0 \in (0,1)$ by

$$i_0 := \frac{1}{2\sqrt{2}} \Phi^{-1}\Big(\frac{\mathcal{E}_0}{4}\Big)\,,$$

so that $\rho_*^2 \geq 8 i_0^2$.

On the one hand it is easy to see, after a change of variable in the integral, using

$$\frac{d}{d\rho}(\mathcal{E}_0 - \Psi(\mathcal{E}_0, \mathcal{J}_0, \rho)) = \frac{2\mathcal{E}_0 \mathcal{J}_0^2}{\rho^3} - 4\Phi'(\rho) \geq \frac{2\mathcal{E}_0 \mathcal{J}_0^2}{\rho^3} \geq 2\mathcal{E}_0 \mathcal{J}_0^2\,,$$

that there holds the bound

$$\tau_* \leq \frac{1}{\mathcal{E}_0 \mathcal{J}_0^2} \int_0^{\mathcal{E}_0(1-\mathcal{J}_0^2)} \frac{dy}{\sqrt{y}} \leq \frac{2\sqrt{1-\mathcal{J}_0^2}}{\mathcal{J}_0^2 \sqrt{\mathcal{E}_0}}\,.$$

So if $\mathcal{J}_0 \geq i_0$, we find that

$$\tau_* \leq \frac{2}{\sqrt{\mathcal{E}_0} i_0^2} = \frac{16}{\sqrt{\mathcal{E}_0}\big(\Phi^{-1}\big(\frac{\mathcal{E}_0}{4}\big)\big)^2}\,.$$

On the other hand for $\mathcal{J}_0 \leq i_0$ we define $\gamma := \Phi^{-1}(\mathcal{E}_0/8)$ and we cut the integral defining τ_* into two parts:

$$\tau_* = \tau_*^{(1)} + \tau_*^{(2)} \quad \text{with} \quad \tau_*^{(1)} = 2\int_{\rho_*}^{\gamma} (\mathcal{E}_0 - \Psi(\mathcal{E}_0, \mathcal{J}_0, \rho))^{-1/2}\, d\rho\,.$$

Notice that since $\rho_*^2 \geq 8 i_0^2$ and $\mathcal{J}_0 \leq i_0$, then $\mathcal{E}_0/4 - \mathcal{E}_0 \mathcal{J}_0^2/4\rho_*^2 \geq 7\mathcal{E}_0/32 \geq \mathcal{E}_0/8$ so

$$\rho_* = \Phi^{-1}\Big(\frac{\mathcal{E}_0}{4} - \frac{\mathcal{E}_0 \mathcal{J}_0^2}{4\rho_*^2}\Big) \leq \Phi^{-1}\Big(\frac{\mathcal{E}_0}{8}\Big) = \gamma\,.$$

The first integral $\tau_*^{(1)}$ is estimated using the fact that Φ' does not vanish outside 1 as stated in Assumption 1.2.1: defining

$$M(\Phi) := \inf_{i_0 \leq \rho \leq \gamma} |\Phi'(\rho)| > 0\,,$$

we find that on $[i_0, \gamma]$,

$$\frac{d}{d\rho}(\mathcal{E}_0 - \Psi(\mathcal{E}_0, \mathcal{J}_0, \rho)) = \frac{2\mathcal{E}_0 \mathcal{J}_0^2}{\rho^3} - 4\Phi'(\rho) \geq 4M(\Phi)$$

so

$$\tau_*^{(1)} \leq \frac{\left(\mathcal{E}_0/2 - \mathcal{E}_0\mathcal{J}_0^2/\gamma^2\right)^{\frac{1}{2}}}{M(\Phi)} \leq \frac{\sqrt{\mathcal{E}_0}}{\sqrt{2}M(\Phi)}\,\cdot$$

For the second integral we estimate simply

$$\tau_*^{(2)} \leq \frac{2}{\left(\mathcal{E}_0/2 - \mathcal{E}_0\mathcal{J}_0^2/\gamma^2\right)^{\frac{1}{2}}} \leq \frac{2}{\left(\mathcal{E}_0/2 - \mathcal{E}_0/8\right)^{\frac{1}{2}}} = \frac{4\sqrt{2}}{\sqrt{3\mathcal{E}_0}}\,\cdot$$

The result follows. $\square$

Remark 3.2.2. — *If Φ is convex then $M(\Phi) = |\Phi'(\gamma)|$. Moreover if Φ is of the type $\frac{1}{\rho^s}\exp(-\frac{1}{1-\rho^2})$ then the proof of Proposition* 3.2.1 *shows that τ_* may be bounded from above by a constant of the order of $C/\sqrt{e_0}(1+\log e_0)$ if $\mathcal{E}_0 \geq e_0$.*

3.3. Scattering cross-section and the Boltzmann collision operator

The scattering operator in Definition 3.2.1 is parametrized by the impact parameter and the two ingoing (or outgoing) velocities. However in the Boltzmann limit the impact parameter makes no longer sense: the observed quantity is the *deflection angle* or *scattering angle*, defined as the angle between ingoing and outgoing relative velocities. The next paragraph defines that angle and as well as the scattering cross-section, and the following paragraph defines the Boltzmann collision operators using that formulation.

3.3.1. Scattering cross-section. — With notation from the previous paragraphs, the deflection angle is equal to $\pi - 2\Theta$ where $\Theta := \alpha + \theta$, the angle α being defined in (3.2.1) and θ being defined in eqreftheta, so that

$$\Theta = \Theta(\mathcal{E}_0, \mathcal{J}_0) := \arcsin \mathcal{J}_0 + \mathcal{J}_0 \int_{\rho_*}^{1} \frac{d\rho}{\sqrt{1 - \frac{4\Phi(\rho)}{\mathcal{E}_0} - \frac{\mathcal{J}_0^2}{\rho^2}}}\,\cdot$$

The following result, and its proof, are due to [**36**]:

Lemma 3.3.1. — *Under Assumption* 1.2.1 *and for all $\mathcal{E}_0 > 0$, the function $\mathcal{J}_0 \mapsto \Theta(\mathcal{E}_0, \mathcal{J}_0) \in [0, \pi/2]$ satisfies $\Theta(\mathcal{E}_0, 0) = 0$ and is strictly monotonic: $\partial_{\mathcal{J}_0}\Theta > 0$ for all $\mathcal{J}_0 \in (0,1)$. Moreover, it satisfies $\lim_{\mathcal{J}_0 \to 0} \partial_{\mathcal{J}_0}\Theta \in (0, \infty]$ and $\lim_{\mathcal{J}_0 \to 1} \partial_{\mathcal{J}_0}\Theta = 0$.*

Proof. — An energy $\mathcal{E}_0 > 0$ being fixed, the limiting values $\Theta(\mathcal{E}_0, 0) = 0$ and $\Theta(\mathcal{E}_0, 1) = \pi/2$ are found by direct computation. To prove monotonicity, the main idea of Saffirio and Simonella is to use the change of variable

$$\sin^2\varphi := \frac{4\Phi(\rho)}{\mathcal{E}_0} + \frac{\mathcal{J}_0^2}{\rho^2}$$

which yields

$$\Theta(\mathcal{E}_0, \mathcal{J}_0) = \arcsin \mathcal{J}_0 + \int_{\arcsin \mathcal{J}_0}^{\frac{\pi}{2}} \frac{\sin\varphi}{\frac{\mathcal{J}_0}{\rho} - \frac{2\rho\Phi'(\rho)}{\mathcal{E}_0\mathcal{J}_0}}\, d\varphi\,.$$

Computing the derivative of this expression with respect to $\mathcal{J}_0$ gives

$$\begin{aligned}\frac{\partial\Theta}{\partial\mathcal{J}_0}(\mathcal{E}_0, \mathcal{J}_0) = {} & \frac{1}{\sqrt{1-\mathcal{J}_0^2}}\left(1 - \frac{\mathcal{E}_0\mathcal{J}_0^2}{\mathcal{E}_0\mathcal{J}_0^2 - \Phi'(1)}\right) \\ & + \int_{\arcsin \mathcal{J}_0}^{\frac{\pi}{2}} \frac{\mathcal{E}_0^2\mathcal{J}_0^2\rho^4\sin\varphi}{(\mathcal{J}_0^2\mathcal{E}_0 - \rho^3\Phi'(\rho))^3}\left(\rho\Phi''(\rho) + 2\Phi'(\rho) + \frac{\rho^3}{\mathcal{E}_0\mathcal{J}_0^2}(\Phi'(\rho))^2\right) d\varphi\end{aligned}$$

where φ is defined by

$$\sin^2\varphi = \frac{\mathcal{J}_0^2}{\rho^2} + \frac{2\Phi(\rho)}{\mathcal{E}_0}\,\cdot$$

In view of the formula giving $\partial_{\mathcal{J}_0}\Theta$, it turns out assumption (1.2.1) implies $\partial_{\mathcal{J}_0}\Theta > 0$ for all $\mathcal{J}_0 \in (0,1)$, and also the limits

$$\lim_{\mathcal{J}_0\to 0} \partial_{\mathcal{J}_0}\Theta \in (0,\infty] \quad \text{and} \lim_{\mathcal{J}_0\to 1} \partial_{\mathcal{J}_0}\Theta = 0$$

as soon as $\Phi'(1) = 0$ (if not then $\lim\limits_{\mathcal{J}_0\to 1} \partial_{\mathcal{J}_0}\Theta = \infty$). The result follows. □

Remark 3.3.2. — *Note that one can construct examples that violate assumption* (1.2.1) *and for which monotonicity fails, regardless of convexity properties of the potential* Φ *(*[**36**]*).*

By Lemma 3.3.1, for each $\mathcal{E}_0$ we can locally invert the map $\Theta(\mathcal{E}_0,\cdot)$, and thus define $\mathcal{J}_0$ as a smooth function of $\mathcal{E}_0$ and Θ. This enables us to define a scattering cross-section (or *collision kernel*), as follows.

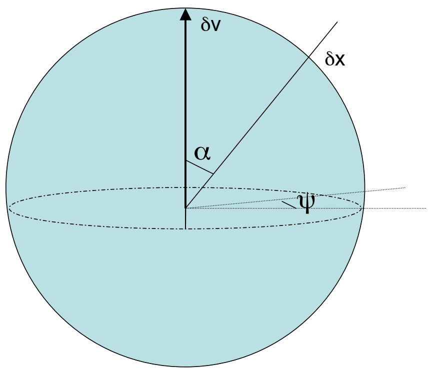

FIGURE 3. Spherical coordinates

For fixed x_1, we denote $d\sigma_1$ the surface measure on the sphere $\{y \in \mathbf{R}^d, |y - x_1| = \varepsilon\}$, to which x_2 belongs. We can parametrize the sphere by (α, ψ), with $\psi \in \mathbf{S}^{d-2}$, where α is the angle defined in (3.2.1). There holds

$$d\sigma_1 = \varepsilon^{d-1}(\sin\alpha)^{d-2} d\alpha d\psi.$$

The direction of the apse line is $\omega = (\Theta, \psi)$, so that, denoting $d\omega$ the surface measure on the unit sphere, there holds

$$d\omega = (\sin\Theta)^{d-2} d\Theta d\psi. \tag{3.3.1}$$

By definition of α in (3.2.1), there holds

$$(x_1 - x_2) \cdot (v_1 - v_2) = \varepsilon |v_1 - v_2| \cos\alpha,$$

so that

$$\begin{aligned}\frac{1}{\varepsilon}\,(x_1 - x_2) \cdot (v_1 - v_2)\, d\sigma_1 &= \varepsilon^{d-1} |v_1 - v_2| \cos\alpha\, (\sin\alpha)^{d-2}\, d\alpha d\psi \\ &= \varepsilon^{d-1} |v_1 - v_2|\, \mathcal{J}_0^{d-2} d\mathcal{J}_0 d\psi\,,\end{aligned}$$

where in the second equality we used the definition of $\mathcal{J}_0$ in (3.2.1). This gives

$$\frac{1}{\varepsilon}\,(x_1 - x_2) \cdot (v_1 - v_2)\, d\sigma_1 = \varepsilon^{d-1} |v_1 - v_2| \mathcal{J}_0^{d-2} \partial_\Theta \mathcal{J}_0\, d\Theta d\psi\,, \tag{3.3.2}$$

wherever $\partial_\Theta \mathcal{J}_0$ is defined, that is, according to Lemma 3.3.1, for $\mathcal{J}_0 \in [0,1)$.

Definition 3.3.3. — *The scattering cross-section is defined for $|v_1 - v_2| > 0$ and $\Theta \in (0, \pi/2]$ by*

$$b(|v_1 - v_2|, \Theta) := |v_1 - v_2| \mathcal{J}_0^{d-2} \partial_\Theta \mathcal{J}_0 (\sin\Theta)^{2-d}\,. \tag{3.3.3}$$

Abusing notation we shall write $b(|v_1 - v_2|, \Theta) = b(|v_1 - v_2|, \omega)$.

By Lemma 3.3.1, the cross-section b is a locally bounded function of the relative velocities and scattering angle.

3.3.2. Scattering cross-section. — The relevance of b is made clear in the derivation of the Boltzmann hierarchy, where we shall use the identity

$$\frac{1}{\varepsilon}\,(x_1 - x_2) \cdot (v_1 - v_2)\, d\sigma_1 = \varepsilon^{d-1} b(|v_1 - v_2|, \omega) d\omega\,, \tag{3.3.4}$$

derived from (3.3.1), (3.3.2) and Definition 3.3.3. As in Chapter 2 (see in particular Paragraph 2.3), we can formally derive the Boltzmann collision operators using this formulation: we thus define

$$\begin{aligned}\mathcal{C}^0_{s,s+1} f^{(s+1)}(t, Z_s) &:= \sum_{i=1}^{s} \int \mathbb{1}_{\nu\cdot(v_{s+1}-v_i)>0}\, \nu \cdot (v_{s+1} - v_i) \\ &\times \Big(f^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i^*, \dots, x_s, v_s, x_i, v_{s+1}^*) - f^{(s+1)}(t, Z_s, x_i, v_{s+1}) \Big) d\nu dv_{s+1}\,,\end{aligned} \tag{3.3.5}$$

where (v_i^*, v_{s+1}^*) is obtained from (v_i, v_{s+1}) by applying the inverse scattering operator σ_0^{-1}, using

$$\sigma_0^{-1}\big(\frac{x_i - x_{s+1}}{|x_i - x_{s+1}|}, v_i, v_{s+1}\big) = \big(\frac{x_i - x_{s+1}}{|x_i - x_{s+1}|}, v_i^*, v_{s+1}^*\big)\,.$$

This can also be written using the cross-section:

$$\begin{aligned}\mathcal{C}^0_{s,s+1} f^{(s+1)}(t, Z_s) &:= \sum_{i=1}^{s} \int b(|v_1 - v_2|, \omega) \\ &\times \Big(f^{(s+1)}(t, x_1, v_1, \dots, x_i, v_i^*, \dots, x_s, v_s, x_i, v_{s+1}^*) - f^{(s+1)}(t, Z_s, x_i, v_{s+1}) \Big) d\omega dv_{s+1}\,.\end{aligned}$$

Remark 3.3.4. — *It is not possible to define an integrable cross-section if the potential is not compactly supported, no matter how fast it might be decaying. This issue is related to the occurrence of grazing collisions and discussed in particular in* [**41**]*, Chapter* 1*, Section* 1.4*. However it is still possible to study the limit towards the Boltzmann equation, if one is ready to change the formulation of the Boltzmann equation by renouncing to the cross-section formulation* ([**36**])*.*

The question of the convergence to Boltzmann in the case of long-range potentials is a challenging open problem; it was considered by Desvillettes and Pulvirenti in [**15**] *in the linear case, while Desvillettes and Ricci studied grazing collisions in* [**16**]*.*

CHAPTER 4

THE BBGKY HIERARCHY

The main goal of this text is to extend the formal strategy described in Chapter 2 for hard spheres to general short-range potentials, then to rigorously justify all the steps of the convergence proof. This necessitates the derivation of bounds for f_N that do not depend on N.

Our starting point is the Liouville equation (1.2.3) satisfied by the N-particle distribution function f_N. We reproduce here equation (1.2.3):

$$\partial_t f_N + \sum_{1\leq i\leq N} v_i \cdot \nabla_{x_i} f_N - \sum_{1\leq i\neq j\leq N} \frac{1}{\varepsilon} \nabla\Phi\left(\frac{x_i - x_j}{\varepsilon}\right) \cdot \nabla_{v_i} f_N = 0\,. \tag{4.0.1}$$

The arguments of f_N in (4.0.1) are $(t, Z_N) \in \mathbf{R}_+ \times \Omega_N$, where

$$\Omega_N := \Big\{ Z_N \in \mathbf{R}^{2dN}\,,\ \forall i \neq j\,,\ x_i \neq x_j \Big\}\,.$$

The classical strategy to obtain a kinetic equation such as (1.3.1) is to write the evolution equation for the first marginal of the distribution function f_N, namely

$$f_N^{(1)}(t, z_1) := \int_{\mathbf{R}^{2d(N-1)}} f_N(t, z_1, z_2, \dots, z_N)\, dz_2 \dots dz_N\,.$$

The point to be noted is that the evolution of $f_N^{(1)}$ depends actually on $f_N^{(2)}$ because of the quadratic interaction imposed by the force $F = -\nabla\Phi$. And in the same way, the equation on $f_N^{(2)}$ depends on $f_N^{(3)}$. Instead of a kinetic equation, we therefore obtain a hierarchy of equations involving all the marginals of f_N

$$f_N^{(s)}(t, Z_s) := \int_{\mathbf{R}^{2d(N-s)}} f_N(t, Z_s, z_{s+1}, \dots, z_N)\, dz_{s+1} \cdots dz_N\,. \tag{4.0.2}$$

In Section 4.1 it is shown that due to the presence of the potential, and contrary to the hard spheres case described in Chapter 2, it is necessary to truncate those marginals away from the set Ω_N. An equation for the *truncated marginals* is derived in weak form in Section 4.2. In order to introduce adequate collision operators, the notion of cluster is introduced and described in Section 4.3. Then collision operators are introduced in Section 4.4, and finally the integral formulation of the equation is written in Section 4.5.

4.1. Truncated marginals

From (4.0.1), we deduce by integration that the *untruncated marginals* defined in (4.0.2) solve

$$\begin{aligned}\partial_t f_N^{(s)}(t,Z_s) + \sum_{i=1}^s v_i\cdot\nabla_{x_i} f_N^{(s)}(t,Z_s) - \frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^s F\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i} f_N^{(s)}(t,Z_s)\\ = \frac{N-s}{\varepsilon}\sum_{i=1}^s\int F\left(\frac{x_i-x_{s+1}}{\varepsilon}\right)\cdot\nabla_{v_i} f_N^{(s+1)}(t,Z_s,z_{s+1})\,dz_{s+1}\,.\end{aligned}\tag{4.1.1}$$

There are several differences between (4.1.1) and the BBGKY hierarchy for hard spheres (2.2.2)-(2.2.3). One is that the transport operator in the left-hand side of (4.1.1) involves a force term. Another is that the integral term in the right-hand side of (4.1.1) involves velocity derivatives. Also, that integral term is a linear integral operator acting on higher-order marginals, just like (2.2.2), but, contrary to (2.2.2), is *not* spatially localized, in the sense that the integral in x_{s+1} is over the whole ball $B(x_i,\varepsilon)$, as opposed to an integral over a sphere in (2.2.2).

This leads us to distinguish spatial configurations in which interactions do take place from spatial configurations in which particles are pairwise at a distance greater than ε, by truncating off the interaction domain $\big\{Z_N,\,|x_i-x_j|\leq\varepsilon \text{ for some } i\neq j\big\}$ in the integrals defining the marginals. For the resulting truncated marginals, collision operators will appear as integrals over a piece of the boundary of the interaction domain, just like in the case of hard spheres. The scattering operator of Chapter 3 (Section 3.2) will then play the role that the boundary condition plays in the case of hard spheres, as sketched in Chapter 2.

Suitable quantities to be studied are therefore not the marginals defined in (4.0.2) but rather the *truncated marginals*

$$\widetilde f_N^{(s)}(t,Z_s) := \int_{\mathbf{R}^{2d(N-s)}} f_N(t,Z_s,z_{s+1},\dots,z_N)\prod_{\substack{i\in\{1,\dots,s\}\\ j\in\{s+1,\dots,N\}}}\mathbb{1}_{|x_i-x_j|>\varepsilon}\,dz_{s+1}\cdots dz_N\,,\tag{4.1.2}$$

where $|\cdot|$ denotes the euclidean norm. Notice that

$$(\widetilde f_N^{(1)} - f_N^{(1)})(t,z_1) = \int_{\mathbf{R}^{2d(N-1)}} f_N(t,z_1,z_2,\dots,z_N)\prod_{j\in\{2,\dots,N\}}(1-\mathbb{1}_{|x_1-x_j|>\varepsilon})\,dz_2\cdots dz_N$$

so that

$$\|(\widetilde f_N^{(1)} - f_N^{(1)})(t)\|_{L^\infty(\mathbf{R}^{2d})} \leq C(N-1)\varepsilon^d\|f_N^{(2)}(t)\|_{L^\infty(\Omega_2)}\,.\tag{4.1.3}$$

We therefore expect both functions to have the same asymptotic behaviour in the Boltzmann-Grad limit $N\varepsilon^{d-1}=1$. This is indeed proved in Lemma 7.1.3 and Corollary 7.2.3 in Chapter 7.

Given $1\leq i<j\leq N$, we denote $dZ_{(i,j)}$ the $2d(j-i+1)$-dimensional Lebesgue measure $dz_i dz_{i+1}\dots dz_j$, and $dX_{(i,j)}$ the $d(j-i+1)$-dimensional Lebesgue measure $dx_i dx_{i+1}\dots dx_j$. We also define

$$\mathcal{D}_N^s := \Big\{X_N\in\mathbf{R}^{dN},\ \forall (i,j)\in[1,s]\times[s+1,N],\ |x_i-x_j|>\varepsilon\Big\}\,,\tag{4.1.4}$$

where $[1,s]$ is short for $[1,s]\cap\mathbf{N}=\{k\in\mathbf{N},\,1\leq k\leq s\}$. Then the truncated marginals (4.1.2) may be formulated as follows:

$$\widetilde f_N^{(s)}(t,Z_s) = \int_{\mathbf{R}^{2d(N-s)}} f_N(t,Z_s,z_{s+1},\dots,z_N)\prod_{\substack{i\in\{1,\dots,s\}\\ j\in\{s+1,\dots,N\}}}\mathbb{1}_{|x_i-x_j|>\varepsilon}\mathbb{1}_{X_N\in\mathcal{D}_N^s}\,dZ_{s+1,N}\,.\tag{4.1.5}$$

The key in introducing the truncated marginals (4.1.5), following King [**27**], is that it allows for a derivation of a hierarchy that is similar to the case of hard spheres. The main drawback is that truncated marginals are not actual *marginals*, in the sense that

$$\widetilde{f}_N^{(s)}(Z_s) \neq \int_{\mathbf{R}^{2d}} \mathbb{1}_B(X_{s+1}) \widetilde{f}_N^{(s+1)}(Z_s, z_{s+1})\, dz_{s+1}\,, \tag{4.1.6}$$

for any $B \subset \mathbf{R}^{d(s+1)}$, in particular if $B = \mathbf{R}^{d(s+1)}$, simply because $\mathcal{D}_N^s$ is *not* included in $\mathcal{D}_N^{s+1}$. Indeed, conditions $|x_j - x_{s+1}| > \varepsilon$, for $j \leq s$, hold for $X_N \in \mathcal{D}_N^s$, but not necessarily for $X_N \in \mathcal{D}_N^{s+1}$. Furthermore, $\mathcal{D}_N^s$ intersects all the $\mathcal{D}_N^{s+m}$, for $m \in [1, N-s]$. A consequence is the existence of higher-order interactions between truncated marginals, as seen below in (4.4.8). Proposition 5.3.1 in Chapter 5 states however that these higher-order interactions are negligible in the Boltzmann-Grad limit.

4.2. Weak formulation of Liouville's equation

Our goal in this section is to find the weak formulation of the system of equations satisfied by the family of truncated marginals $\big(\widetilde{f}_N^{(s)}\big)_{s\in[1,N]}$ defined above in (4.1.5). From now on we assume that f_N decays at infinity in the velocity variable.

Given a smooth, compactly supported function ϕ defined on $\mathbf{R}_+ \times \mathbf{R}^{2ds}$ and satisfying the symmetry assumption (1.1.1), we have

$$\begin{aligned}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} \Big(\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N - \frac{1}{\varepsilon} \sum_{i=1}^N \sum_{j\neq i} F\left(\frac{x_i - x_j}{\varepsilon}\right) \cdot \nabla_{v_i} f_N\Big)(t, Z_N) \\ \times \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N dt = 0\,.\end{aligned} \tag{4.2.1}$$

Note that in the above double sum in i and j, all the terms vanish except when $(i,j) \in [1,s]^2$ and when $(i,j) \in [s+1,N]^2$, by assumption on the support of F.

We now use integrations by parts to derive from (4.2.1) the weak form of the equation in the marginals $\widetilde{f}_N^{(s)}$. On the one hand an integration by parts in the time variable gives

$$\begin{aligned}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} \partial_t f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N dt = &- \int_{\mathbf{R}^{2dN}} f_N(0, Z_N) \phi(0, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N \\ &- \int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} f_N(t, Z_N) \partial_t \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N dt\,,\end{aligned}$$

hence, by definition of $\widetilde{f}_N^{(s)}$,

$$\begin{aligned}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} \partial_t f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N dt = &- \int_{\mathbf{R}^{2ds}} \widetilde{f}_N^{(s)}(0, Z_s) \phi(0, Z_s)\, dZ_s \\ &- \int_{\mathbf{R}_+\times\mathbf{R}^{2ds}} \widetilde{f}_N^{(s)}(t, Z_s) \partial_t \phi(t, Z_s)\, dZ_s dt\,.\end{aligned}$$

Now let us compute

$$\sum_{i=1}^N \int_{\mathbf{R}^{2dN}} v_i \cdot \nabla_{x_i} f_N(t, Z_N) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N = \int_{\mathbf{R}^{2dN}} \mathrm{div}_{X_N} \big(V_N\, f_N(t, Z_N)\big) \phi(t, Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s}\, dZ_N$$

using Green's formula. The boundary of $\mathcal{D}_N^s$ is made of configurations with at least one pair (i,j), satisfying $1 \leq i \leq s$ and $s+1 \leq j \leq N$, with $|x_i - x_j| = \varepsilon$.

Let us define, for any couple $(i,j) \in [1,N]^2$,

$$\Sigma_N^s(i,j) := \Big\{ Z_N \in \mathbf{R}^{2dN}, \quad |x_i - x_j| = \varepsilon \\ \text{and} \quad \forall (k,\ell) \in [1,s] \times [s+1,N] \setminus \{i,j\},\ |x_k - x_\ell| > \varepsilon \Big\}. \tag{4.2.2}$$

We notice that $\Sigma_N^s(i,j)$ is a submanifold of $\big\{Z_N \in \mathbf{R}^{2dN},\, |x_i - x_j| = \varepsilon\big\}$, which is a smooth, codimension 1 manifold of $\mathbf{R}^{2dN}$ (locally isomorphic to the space $\mathbf{S}_\varepsilon^d \times \mathbf{R}^{d(2N-1)}$), and we denote by $d\sigma_N^{i,j}$ its surface measure, induced by the Lebesgue measure. Configurations with more than one *collisional* pair, i.e., (i,j) and (i',j') with $1 \leq i,i' \leq s$, $s+1 \leq j,j' \leq N$, with $|x_i - x_j| = |x_{i'} - x_{j'}| = \varepsilon$, and $\{i,j\} \neq \{i',j'\}$, are subsets of submanifols of $\mathbf{R}^{dN}$ of dimension at least two, and therefore contribute nothing to the boundary terms.

Denoting $\nu^{i,j} := \dfrac{x_i - x_j}{|x_i - x_j|}$ we therefore obtain by Green's formula:

$$\begin{aligned}
&\sum_{i=1}^N \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} v_i \cdot \nabla_{x_i} f_N(t,Z_N) \phi(t,Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, dZ_N \, dt \\
&\qquad = -\sum_{i=1}^s \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} f_N(t,Z_N) v_i \cdot \nabla_{x_i} \phi(t,Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, dZ_N dt \\
&\qquad\quad + \frac{1}{\sqrt{2}} \sum_{i=1}^s \sum_{j=s+1}^N \int_{\mathbf{R}_+ \times \Sigma_N^s(i,j)} \nu^{i,j} \cdot (v_j - v_i) \, f_N(t,Z_N) \phi(t,Z_s) \, d\sigma_N^{i,j} dt \,.
\end{aligned}$$

By symmetry (1.1.1), this gives

$$\begin{aligned}
&\sum_{i=1}^N \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} v_i \cdot \nabla_{x_i} f_N(t,Z_N) \phi(t,Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, dZ_N \, dt \\
&\qquad = -\sum_{i=1}^s \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} f_N(t,Z_N) v_i \cdot \nabla_{x_i} \phi(t,Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, dZ_N dt \\
&\qquad\quad + \frac{1}{\sqrt{2}} (N-s) \sum_{i=1}^s \int_{\mathbf{R}_+ \times \Sigma_N^s(i,s+1)} \nu^{i,s+1} \cdot (v_{s+1} - v_i) \, f_N(t,Z_N) \phi(t,Z_s) \, d\sigma_N^{i,s+1} dt \,,
\end{aligned}$$

so finally by definition of $\widetilde{f}_N^{(s)}$, we obtain

$$\begin{aligned}
&\sum_{i=1}^N \int_{\mathbf{R}_+ \times \mathbf{R}^{2dN}} v_i \cdot \nabla_{x_i} f_N(t,Z_N) \phi(t,Z_s) \mathbb{1}_{X_N \in \mathcal{D}_N^s} \, dZ_N \, dt \\
&\qquad = -\sum_{i=1}^s \int_{\mathbf{R}_+ \times \mathbf{R}^{2ds}} \widetilde{f}_N^{(s)}(t,Z_s) v_i \cdot \nabla_{x_i} \phi(t,Z_s) \, dZ_s dt \\
&\qquad\quad + \frac{1}{\sqrt{2}} (N-s) \sum_{i=1}^s \int_{\mathbf{R}_+ \times \Sigma_N^s(i,s+1)} \nu^{i,s+1} \cdot (v_{s+1} - v_i) \, f_N(t,Z_N) \phi(t,Z_s) \, d\sigma_N^{i,s+1} dt \,.
\end{aligned} \tag{4.2.3}$$

Now let us consider the contribution of the potential in (4.2.1). We split the sum as follows:

$$\begin{aligned}
&\frac{1}{\varepsilon}\sum_i\sum_{j\neq i}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} F\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i}f_N(t,Z_N)\phi(t,Z_s)\mathbb{1}_{X_N\in\mathcal{D}_N^s}\,dZ_Ndt\\
&\quad=\frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ j\neq i}}^{s}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} F\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i}f_N(t,Z_N)\phi(t,Z_s)\mathbb{1}_{X_N\in\mathcal{D}_N^s}\,dZ_Ndt\\
&\qquad+\frac{1}{\varepsilon}\sum_{\substack{i,j=s+1\\ j\neq i}}^{N}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} F\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i}f_N(t,Z_N)\phi(t,Z_s)\mathbb{1}_{X_N\in\mathcal{D}_N^s}\,dZ_Ndt\,.
\end{aligned}$$

We notice that the second term in the right-hand side vanishes identically. It follows that

$$\begin{aligned}
&\frac{1}{\varepsilon}\sum_i\sum_{j\neq i}\int_{\mathbf{R}_+\times\mathbf{R}^{2dN}} F\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i}f_N(t,Z_N)\phi(t,Z_s)\mathbb{1}_{X_N\in\mathcal{D}_N^s}\,dZ_Ndt\\
&\quad=-\frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ j\neq i}}^{s}\int_{\mathbf{R}_+\times\mathbf{R}^{2ds}} F\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i}\phi(t,Z_s)\widetilde{f}_N^{(s)}(t,Z_s)\,dZ_sdt
\end{aligned}$$

so in the end we obtain

$$\begin{aligned}
&\int_{\mathbf{R}_+\times\mathbf{R}^{2ds}}\widetilde{f}_N^{(s)}(t,Z_s)\Big(\partial_t\phi+\mathrm{div}_{X_s}\,(V_s\phi)+\frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ j\neq i}}^{s}F\left(\frac{x_i-x_j}{\varepsilon}\right)\cdot\nabla_{v_i}\phi\Big)(t,Z_s)\,dZ_sdt\\
&\quad=-\int_{\mathbf{R}^{2ds}}\widetilde{f}_N^{(s)}(0,Z_s)\phi(0,Z_s)\,dZ_s\\
&\qquad-\frac{N-s}{\sqrt{2}}\sum_{i=1}^{s}\int_{\mathbf{R}_+\times\Sigma_N^s(i,s+1)}\nu^{i,s+1}\cdot(v_{s+1}-v_i)\,f_N(t,Z_N)\phi(t,Z_s)\,d\sigma_N^{i,s+1}dt\,.
\end{aligned}\tag{4.2.4}$$

Remark 4.2.1. — *Using the weak form of Liouville's equation, we see that multiple collisions (which occur as a boundary integral on a zero measure subset of $\partial\mathcal{D}_N^s$) can be neglected.*

4.3. Clusters

We want to analyze the second term on the right-hand side of (4.2.4). We notice that in the space-velocity integration the variables $z_{s+2},\dots,z_N$ are integrated over $\mathbf{R}^{d(N-s-1)}$ (with the restriction that they must be at a distance at least ε from X_s) whereas z_{s+1} must lie in the sphere centered at x_i and of radius ε. It is therefore natural to try to express that contribution in terms of the marginal $\widetilde{f}_N^{(s+1)}(Z_{s+1})$. However as pointed out in (4.1.6),

$$\int\widetilde{f}_N^{(s+1)}(Z_{s+1})\,dz_{s+1}\neq\widetilde{f}_N^{(s)}(Z_s)\,.$$

The difference between those two terms is that on the one hand

$$\forall X_N\in\mathcal{D}_N^{s+1}\,,\qquad\text{one has }|x_j-x_{s+1}|>\varepsilon\text{ for all }j\geq s+2\,,$$

which is not the case for $X_N\in\mathcal{D}_N^s$, and on the other hand

$$\forall X_N\in\mathcal{D}_N^s\,,\qquad\text{one has }|x_j-x_{s+1}|>\varepsilon\text{ for all }j\leq s\,,$$

a condition which does not appear in the definition of $\mathcal{D}_N^{s+1}$.

This leads to the following definition.

***Definition 4.3.1* (ε-closure).** — *Given a subset $X_N = \{x_1, \ldots, x_N\}$ of $\mathbf{R}^{dN}$ and an integer s in $[1, N]$, the ε-closure $E(X_s, X_N)$ of X_s in X_N is defined as the intersection of all subsets Y of X_N which contain X_s and satisfy the separation condition*

$$\forall y \in Y\,, \quad \forall x \in X_N \setminus Y\,, \quad |x - y| > \varepsilon\,. \tag{4.3.1}$$

We denote $|E(X_s, X_N)|$ the cardinal of $E(X_s, X_N)$.

Now let us introduce the following notation, useful in situations where X_N belongs to $\Sigma_N^s(i, s+1)$, defined in (4.2.2)

Notation 4.3.2. — *If $X_{s+m} = E(X_s, X_{s+m})$ and if for some integers $j_0 \leq s < k_0 \leq s+m$, there holds $|x_j - x_k| > \varepsilon$ for all $(j, k) \in [1, s] \times [s+1, s+m] \setminus \{(j_0, k_0)\}$, then we say that $E(X_s, X_{s+m})$ has a weak link at (j_0, k_0), and we denote $X_{s+m} = E_{\langle j_0, k_0 \rangle}(X_s, X_{s+m})$.*

Moreover the following notion, following King [**27**], will turn out to be very useful.

***Definition 4.3.3* (Cluster).** — *A cluster of base $X_s = \{x_1, \ldots, x_s\}$ and length m is any point $\{x_{s+1}, \ldots, x_{s+m}\}$ in $\mathbf{R}^{dm}$ such that $E(X_s, X_{s+m}) = X_{s+m}$. We denote $\Delta_m(X_s)$ the set of all such clusters.*

The proof of the following lemma is completely elementary.

Lemma 4.3.4. — *The following equivalences hold, for $m \geq 1$:*

$$\Big(E(X_s, X_N) = X_{s+m}\Big) \iff \Big(E(X_s, X_{s+m}) = X_{s+m} \text{ and } X_N \in \mathcal{D}_N^{s+m}\Big)\,, \tag{4.3.2}$$

$$\begin{pmatrix} E(X_s, X_N) = X_{s+m} \\ X_N \in \Sigma_N^s(i, s+1) \end{pmatrix} \iff \begin{pmatrix} E_{\langle i, s+1 \rangle}(X_s, X_{s+m}) = X_{s+m} \\ X_N \in \mathcal{D}_N^{s+m} \\ |x_i - x_{s+1}| = \varepsilon \end{pmatrix}\,, \tag{4.3.3}$$

as well as the implication, for $m \geq 2$,

$$\Big(E_{\langle i, s+1 \rangle}(X_s, X_{s+m}) = X_{s+m}\Big) \implies \Big(\big\{x_{s+2}, \ldots, x_{s+m}\big\} \in \Delta_{m-1}(x_{s+1})\Big)\,. \tag{4.3.4}$$

4.4. Collision operators

With the help of the notions introduced in Section 4.3, we now can reformulate the boundary integral in (4.2.4).

Given $1 \leq s \leq N-1$ and X_N in $\Sigma_N^s(i, s+1)$, there holds $|x_{s+1} - x_i| = \varepsilon$, so that x_{s+1} belongs to $E(X_s, X_N)$, implying $|E(X_s, X_N)| \geq s+1$. We decompose $\Sigma_N^s(i, s+1)$ into a disjoint union over the possible cardinals of the ε-closure of X_s in X_N :

$$\Sigma_N^s(i, s+1) = \bigcup_{1 \leq m \leq N-s} \Big(\Sigma_N^s(i, s+1) \bigcap \big\{Y_N,\ |E(Y_s, Y_N)| = s+m\big\}\Big)\,, \tag{4.4.1}$$

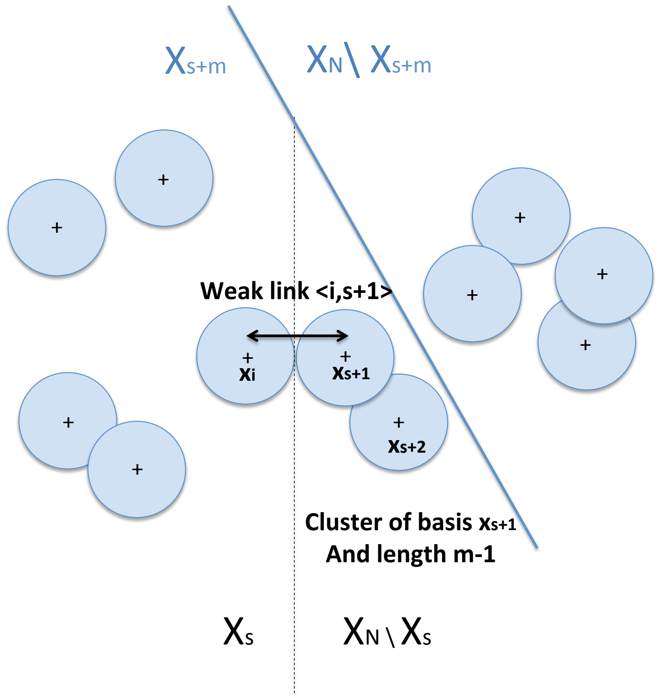

FIGURE 4. Clusters with weak links

implying

$$\int_{\Sigma_N^s(i,s+1)} \nu^{i,s+1}\cdot(v_{s+1}-v_i)\, f_N(Z_N)\phi(Z_s)\, d\sigma_N^{i,s+1}$$
$$= \sum_{1\le m\le N-s} \int_{\Sigma_N^s(i,s+1)} 1\!\!1_{|E(X_s,X_N)|=s+m}\, \nu^{i,s+1}\cdot(v_{s+1}-v_i)\, f_N(Z_N)\phi(Z_s)\, d\sigma_N^{i,s+1}\,.$$

By assumption of symmetry (1.1.1) for f_N and ϕ, if $|E(X_s,X_N)|=s+m$, we can index the particles so that $E(X_s,X_N)=X_{s+m}$: we obtain

$$\begin{aligned}&\int_{\Sigma_N^s(i,s+1)} 1\!\!1_{|E(X_s,X_N)|=s+m}\, \nu^{i,s+1}\cdot(v_{s+1}-v_i)\, f_N(Z_N)\phi(Z_s)\, d\sigma_N^{i,s+1}\\ &\qquad = C_{N-s-1}^{m-1}\int_{\Sigma_N^s(i,s+1)} 1\!\!1_{E(X_s,X_N)=X_{s+m}}\, \nu^{i,s+1}\cdot(v_{s+1}-v_i)\, f_N(Z_N)\phi(Z_s)\, d\sigma_N^{i,s+1}\,.\end{aligned}\tag{4.4.2}$$

We use equivalence (4.3.3) from Lemma 4.3.4 and Fubini's theorem to write

$$\begin{aligned}&\int_{\Sigma_N^s(i,s+1)} 1\!\!1_{E(X_s,X_N)=X_{s+m}} \nu^{i,s+1}\cdot(v_{s+1}-v_i) f_N(Z_N)\phi(Z_s) d\sigma_N^{i,s+1}\\ &= \sqrt{2}\int_{S_\varepsilon(x_i)\times\mathbf{R}^d} \nu^{i,s+1}\cdot(v_{s+1}-v_i)\phi(Z_s)\\ &\qquad\times\Big(\int_{\mathbf{R}^{2d(m-1)}} 1\!\!1_{E_{\langle i,s+1\rangle}(X_s,X_{s+m})=X_{s+m}} f_N^{(s+m)}(Z_{s+m})dZ_{(s+1,s+m)}\Big)\, d\sigma_i(x_{s+1})\,,\end{aligned}$$

with $d\sigma_i$ the surface measure on $S_\varepsilon(x_i):=\big\{x\in\mathbf{R}^d,\ |x-x_i|=\varepsilon\big\}$. With (4.3.4), if $m\ge 2$, then the above integral over $\mathbf{R}^{2d(m-1)}$ appears as an integral over $\Delta_{m-1}(x_{s+1})$. We also remark that in the case $m = 1$, we have a simple description of $E_{\langle i,s+1\rangle}(X_s, X_{s+1}) = X_{s+1}$:

$$\left(\mathbb{1}_{E_{\langle i,s+1\rangle}(X_s,X_{s+1})=X_{s+1}} \neq 0\right) \iff \begin{pmatrix} |x_i - x_{s+1}| \leq \varepsilon \\ |x_j - x_{s+1}| > \varepsilon \quad \text{for } j \in [1,s] \setminus \{i\} \end{pmatrix}. \tag{4.4.3}$$

This leads to the following definition of the collision term of order $m \geq 1$, for $s + m \leq N$: we define

$$\begin{aligned} \mathcal{C}_{s,s+m} \widetilde{f}_N^{(s+m)}(Z_s) := m C_{N-s}^m \sum_{i=1}^s \int_{S_\varepsilon(x_i)\times \mathbf{R}^d} & \nu^{s+1,i} \cdot (v_{s+1} - v_i) \\ & \times G^{(m-1)}_{\langle i,s+1\rangle}(f_N^{(s+m)})(Z_{s+1})\, d\sigma_i(x_{s+1}) dv_{s+1}\,, \end{aligned} \tag{4.4.4}$$

where for $m = 1$, by (4.4.3):

$$G^{(0)}_{\langle i,s+1\rangle}(\widetilde{f}_N^{(s+1)})(Z_{s+1}) := \Big(\prod_{\substack{1\leq j\leq s\\ j\neq i}} \mathbb{1}_{|x_{s+1}-x_j|>\varepsilon}\Big) \widetilde{f}_N^{(s+1)}(Z_{s+1})\,, \tag{4.4.5}$$

and for $m \geq 2$:

$$\begin{aligned} & G^{(m-1)}_{\langle i,s+1\rangle}(\widetilde{f}_N^{(s+m)})(Z_{s+1}) \\ & \qquad := \int_{\Delta_{m-1}(x_{s+1})\times \mathbf{R}^{d(m-1)}} \mathbb{1}_{E_{\langle i,s+1\rangle}(X_s,X_{s+m})=X_{s+m}} \widetilde{f}_N^{(s+m)}(Z_{s+m}) dZ_{(s+2,s+m)}\,. \end{aligned} \tag{4.4.6}$$

The complex-looking indicator function $\mathbb{1}_{E_{\langle i,s+1\rangle}(X_s,X_{s+m})=X_{s+m}}$ will, in the estimates of the next chapters, be simply bounded from above by one. This will be the case for instance in an estimate showing that higher-order collision operators (4.4.6) are negligible in the thermodynamical limit; this estimate is (5.3.1) in Proposition 5.3.1.

With $(N - s)C^{m-1}_{N-s-1} = mC^m_{N-s}$, we can now reformulate (4.2.4) into

$$\begin{aligned} & \int_{\mathbf{R}_+\times \mathbf{R}^{2ds}} \widetilde{f}_N^{(s)}(t, Z_s)\Big(\partial_t \phi + \operatorname{div}_{X_s}(V_s\phi) - \frac{1}{\varepsilon}\sum_{\substack{i,j=1\\ j\neq i}}^{s} \nabla\Phi\left(\frac{x_i - x_j}{\varepsilon}\right)\cdot \nabla_{v_i}\phi\Big)(t, Z_s)\, dZ_s dt \\ & \qquad + \int_{\mathbf{R}^{2ds}} \widetilde{f}_N^{(s)}(0, Z_s)\phi(0, Z_s)\, dZ_s = \sum_{m=1}^{N-s} \int_{\mathbf{R}^+\times \mathbf{R}^{2ds}} \phi(t, Z_s)\mathcal{C}_{s,s+m}\widetilde{f}_N^{(s+m)}(t, Z_s)\, dt dZ_s\,, \end{aligned} \tag{4.4.7}$$

so that $\widetilde{f}_N^{(s)}$ appears as a (formal) weak solution to

$$\partial_t \widetilde{f}_N^{(s)} + \sum_{1\leq i\leq s} v_i \cdot \nabla_{x_i}\widetilde{f}_N^{(s)} - \frac{1}{\varepsilon}\sum_{1\leq i\neq j\leq s} \nabla\Phi\left(\frac{x_i - x_j}{\varepsilon}\right)\cdot \nabla_{v_i}\widetilde{f}_N^{(s)} = \sum_{m=1}^{N-s} \mathcal{C}_{s,s+m}\widetilde{f}_N^{(s+m)}\,. \tag{4.4.8}$$

4.5. Mild solutions

We now define the integral formulation of (4.4.8). Denote by $\mathbf{\Phi}_s(t)$ the s-particle Hamiltonian flow, and by $\mathbf{H}_s$ the associated solution operator:

$$\mathbf{H}_s(t): \qquad f \in C^0(\Omega_s;\mathbf{R}) \mapsto f(\mathbf{\Phi}_s(-t,\cdot)) \in C^0(\Omega_s;\mathbf{R})\,. \tag{4.5.1}$$

The time-integrated form of equation (4.4.8) is

$$\widetilde{f}_N^{(s)}(t, Z_s) = \mathbf{H}_s(t)\widetilde{f}_N^{(s)}(0, Z_s) + \sum_{m=1}^{N-s}\int_0^t \mathbf{H}_s(t-\tau)\mathcal{C}_{s,s+m}\widetilde{f}_N^{(s+m)}(\tau, Z_s)\, d\tau\,. \tag{4.5.2}$$

The *total flow* and *total collision* operators $\mathbf{H}$ and $\mathbf{C}_N$ are defined on finite sequences $G_N = (g_s)_{1\leq s\leq N}$ as follows:

$$
\begin{cases} \forall s \leq N\,,\ \big(\mathbf{H}(t)G_N\big)_s := \mathbf{H}_s(t)g_s\,, \\ \forall\, s \leq N-1\,,\ \big(\mathbf{C}_N G_N\big)_s := \displaystyle\sum_{m=1}^{N-s} \mathcal{C}_{s,s+m}\, g_{s+m}\,, \quad \big(\mathbf{C}_N G_N\big)_N := 0\,. \end{cases} \tag{4.5.3}
$$

We define *mild solutions* to the BBGKY hierarchy (4.5.2) to be solutions of

$$
\widetilde{F}_N(t) = \mathbf{H}(t)\widetilde{F}_N(0) + \int_0^t \mathbf{H}(t-\tau)\mathbf{C}_N \widetilde{F}_N(\tau)\, d\tau\,, \qquad \widetilde{F}_N = (\widetilde{f}_N^{(s)})_{1\leq s\leq N}\,. \tag{4.5.4}
$$

Remark 4.5.1. — *At this stage, the use of weak formulations could seem a little bit suspicious since they are used essentially as a technical artifice to go from the Liouville equation* (1.2.3) *to the mild form of the BBGKY hierarchy* (4.5.2). *In particular, this allows to ignore pathological trajectories involving multiple collisions. Nevertheless, the existence of mild solutions to the BBGKY hierarchy (to be proved in the next two chapters) provides the existence of weak solutions to the BBGKY hierarchy, and in particular to the Liouville equation (which is nothing else than the last equation of the hierarchy). The classical uniqueness result for kinetic transport equations then implies that the object we consider, that is the family of truncated marginals, is uniquely determined (almost everywhere).*

Note that similarly we can define the total Boltzmann flow and collision operators $\mathbf{S}$ and $\mathbf{C}$ as follows:

$$
\begin{cases} \forall s \geq 1\,,\ \big(\mathbf{S}(t)G\big)_s := \mathbf{S}_s(t)g_s\,, \\ \forall\, s \geq 1\,,\ \big(\mathbf{C^0}G\big)_s := \mathcal{C}^0_{s,s+1}\, g_{s+1}\,, \end{cases} \tag{4.5.5}
$$

where $\mathbf{S}_s$ denotes the free transport operator in s-particle space and $\mathcal{C}^0_{s,s+1}$ is defined in (3.3.5).

CHAPTER 5

CONTINUITY OF COLLISION OPERATORS

In view of proving the existence of mild solutions to the BBGKY hierarchy (4.5.2), we need continuity estimates on the linear collision operators $\mathcal{C}_{s,s+m}$ defined in (4.4.4)-(4.4.5)-(4.4.6), and the total collision operator $\mathbf{C}_N$ defined in (4.5.3).

We first note that, by definition, the operator $\mathcal{C}_{s,s+m}$ involves only configurations with clusters of length m. Classical computations of statistical mechanics, presented in Section 5.1, show that the probability of finding such clusters is exponentially decreasing with m.

It is then natural to introduce functional spaces encoding the decay with respect to energy and the growth with respect to the order of the marginal (see Section 5.2). In these appropriate functional spaces, we can establish uniform continuity estimates for the BBGKY (Section 5.3) as well as for the limiting Boltzmann collision operators (Section 5.4).

5.1. Cluster estimates

A point $X_s \in \mathbf{R}^{ds}$ being given, we recall that $\Delta_m(X_s)$ is the set of all clusters of base X_s and length m (this notation is introduced in Definition 4.3.3 page 26).

Lemma 5.1.1. — *For any symmetric function φ on $\mathbf{R}^{Nd}$, any $s \in [1, N-1]$, any $X_s \in \mathbf{R}^{ds}$, the following identity holds:*

$$\begin{aligned}\int_{\mathbf{R}^{(N-s)d}} \varphi(X_N) dX_{(s+1,N)} &= \int_{\mathbf{R}^{d(N-s)}} 1\!\!1_{X_N \in \mathcal{D}_N^s} \, \varphi(X_N) \, dX_{(s+1,N)} \\ &+ \sum_{m=1}^{N-s} C_{N-s}^m \int_{\Delta_m(X_s)} \left(\int_{\mathbf{R}^{d(N-s-m)}} 1\!\!1_{X_N \in \mathcal{D}_N^{s+m}} \, \varphi(X_N) \, dX_{(s+m+1,N)} \right) dX_{(s+1,s+m)} \,, \end{aligned} \tag{5.1.1}$$

implying, for $\zeta > 0$,

$$\frac{1}{m!} \int_{\Delta_m(X_s)} dX_{(s+1,s+m)} \leq \zeta^{-m} \exp\big(\zeta \kappa_d (s+m) \varepsilon^d\big) \tag{5.1.2}$$

and

$$\sum_{m \geq 1} \frac{\zeta^{m+1} \exp\big(- \zeta \kappa_d (m+1) \varepsilon^d \big)}{m!} \int_{\Delta_m(x_1)} dX_{(2,m+1)} \leq \zeta \big(1 - \exp\big(- \zeta \kappa_d \varepsilon^d \big)\big) \,, \tag{5.1.3}$$

where κ_d is the volume of the unit ball in $\mathbf{R}^d$.

Proof. — The first identity (5.1.1) is obtained by a simple partitioning argument, which extends the splitting used to define $\mathcal{C}_{s,s+m}$ in (4.4.4) in the previous chapter. We recall that, given any $X_s \in \mathbf{R}^{ds}$, the family

$$\Big\{(x_{s+1},\dots,x_N)\,,\,|E(X_s,X_N)| = s+m\Big\} \quad \text{for } 0\le m\le N-s\,,$$

is a partition of $\mathbf{R}^{(N-s)d}$. Then we use the symmetry assumption, as we did in (4.4.2), to find

$$\int_{\mathbf{R}^{(N-s)d}} \varphi(X_N)dX_{(s+1,N)} = \sum_{0\le m\le N-s} C^m_{N-s}\int_{\mathbf{R}^{(N-s)d}} 1\!\!1_{E(X_s,X_N)=X_{s+m}}\varphi(X_N)dX_{(s+1,N)}\,.$$

It then suffices to use equivalence (4.3.2) from Lemma 4.3.4, noting that the set of all $(x_{s+1},\dots,x_{s+m})$ in $\mathbf{R}^{md}$ such that $E(X_s,X_{s+m}) = X_{s+m}$ coincides with $\Delta_m(X_s)$. This proves (5.1.1).

Estimates (5.1.2) and (5.1.3) come from the counterpart of (5.1.1) at the grand canonical level, i.e. when the activity ζ^{-1} is fixed, rather than the total number N of particles; Remark 5.2.3 expands on this terminology.

For any bounded $\Lambda\subset\mathbf{R}^d$, the associated grand-canonical ensemble for n non-interacting particles is defined as the probability measure with density

$$\varphi_n(X_n) := \frac{\zeta^n\exp(-\zeta|\Lambda|)}{n!}\prod_{1\le i\le n} 1\!\!1_{x_i\in\Lambda}\,.$$

The s-point correlation function g_s and the truncated s-point correlation function $\widetilde g_s$ are defined by

$$\begin{aligned}
g_s(X_s) &:= \sum_{n=s}^{\infty}\frac{n!}{(n-s)!}\int_{\mathbf{R}^{(n-s)d}}\varphi_n(X_n)dX_{(s+1,n)}\,,\\
\widetilde g_s(X_s) &:= \sum_{n=s}^{\infty}\frac{n!}{(n-s)!}\int_{\mathbf{R}^{(n-s)d}} 1\!\!1_{X_n\in\mathcal{D}^s_n}\varphi_n(X_n)dX_{(s+1,n)}\,.
\end{aligned}$$

We compute

$$\int_{\mathbf{R}^{(n-s)d}}\varphi_n(X_n)dX_{(s+1,n)} = \zeta^s\exp\big(-\zeta|\Lambda|\big)\frac{(\zeta|\Lambda|)^{n-s}}{n!}\prod_{1\le i\le s}1\!\!1_{x_i\in\Lambda}\,,$$

so that

$$g_s(X_s) = \zeta^s\exp\big(-\zeta|\Lambda|\big)\sum_{k=0}^{\infty}\frac{(\zeta|\Lambda|)^k}{k!}\prod_{1\le i\le s}1\!\!1_{\Lambda}(x_i) = \zeta^s\prod_{1\le i\le s}1\!\!1_{x_i\in\Lambda}\,. \tag{5.1.4}$$

Similarly, by definition of $\mathcal{D}^s_n$ in (4.1.4),

$$\int_{\mathbf{R}^{(n-s)d}}1\!\!1_{X_n\in\mathcal{D}^s_n}\prod_{s+1\le j\le n}1\!\!1_{x_i\in\Lambda}\,dX_{(s+1,n)} = \Big|\Lambda\cap\Big(\bigcap_{1\le i\le s}{}^cB_\varepsilon(x_i)\Big)\Big| = \big|\Lambda\cap{}^cB_\varepsilon(X_s)\big|\,,$$

where we denote $B_\varepsilon(X_s) := \bigcup_{1\le i\le s}B_\varepsilon(x_i)$, with $B_\varepsilon(x_i) := \big\{y\in\mathbf{R}^d,\ |y-x_i|\le\varepsilon\big\}$. This implies

$$\widetilde g_s(X_s) = \zeta^s\exp\big(-\zeta|\Lambda|\big)\sum_{n\ge s}\frac{\big(\zeta|\Lambda\cap{}^cB_\varepsilon(X_s)|\big)^{n-s}}{(n-s)!}\prod_{1\le i\le s}1\!\!1_{x_i\in\Lambda}\,.$$

If $B_\varepsilon(x_i)\subset\Lambda$ for all $1\le i\le s$, then $|\Lambda| - |\Lambda\cap{}^cB_\varepsilon(X_s)| = |B_\varepsilon(X_s)|$. We obtain

$$\widetilde g_s(X_s) = \zeta^s\exp\big(-\zeta|B_\varepsilon(X_s)|\big)\,,\qquad \text{if } B_\varepsilon(X_s)\subset\Lambda\,. \tag{5.1.5}$$

Besides, by (5.1.1),

$$g_s(X_s) = \widetilde{g}_s(X_s) + \sum_{n=s}^{\infty}\sum_{m=1}^{n-s} \frac{n!C_{n-s}^m}{(n-s)!} \int_{\Delta_m(X_s)} \Big(\int_{\mathbf{R}^{(n-s-m)d}} 1\!\!1_{X_n \in \mathcal{D}_n^{s+m}} g_s(X_n)\, dX_{(s+m+1,n)} \Big)\, dX_{(s+1,s+m)} \, .$$

By Fubini, we get

$$\begin{aligned}
&\sum_{n=s}^{\infty}\sum_{m=1}^{n-s} \frac{n!C_{n-s}^m}{(n-s)!} \int_{\Delta_m(X_s)} \Big(\int_{\mathbf{R}^{(n-s-m)d}} 1\!\!1_{X_n \in \mathcal{D}_n^{s+m}} \varphi_n(X_n)\, dX_{(s+m+1,n)} \Big)\, dX_{(s+1,s+m)} \\
&\quad = \sum_{n=s}^{\infty}\sum_{m=1}^{n-s} \frac{n!}{(k-s)!(n-k)!} \int_{\Delta_{k-s}(X_s)} \Big(\int_{\mathbf{R}^{(n-k)d}} 1\!\!1_{X_n \in \mathcal{D}_n^{k}} \varphi_n(X_n)\, dX_{(k+1,n)} \Big)\, dX_{(s+1,k)} \\
&\quad = \sum_{k=s+1}^{\infty} \frac{1}{(k-s)!} \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} \int_{\Delta_{k-s}(X_s)} \Big(\int_{\mathbf{R}^{(n-k)d}} 1\!\!1_{X_n \in \mathcal{D}_n^{k}} \varphi_n(X_n)\, dX_{(k+1,n)} \Big)\, dX_{(s+1,k)} \\
&\quad = \sum_{k=s+1}^{\infty} \frac{1}{(k-s)!} \int_{\Delta_{k-s}(X_s)} \widetilde{g}_k(X_k) dX_{(s+1,k)} \, .
\end{aligned}$$

We have proved that

$$g_s(X_s) = \widetilde{g}_s(X_s) + \sum_{k=s+1}^{\infty} \frac{1}{(k-s)!} \int_{\Delta_{k-s}(X_s)} g_k(X_k) dX_{(s+1,k)} \, . \tag{5.1.6}$$

We now show how identities (5.1.4)-(5.1.5)-(5.1.6) imply the bounds (5.1.2)-(5.1.3).

We first retain only the contribution of $k = s+m$ in the right-hand side of (5.1.6). Given $\varepsilon > 0$ and $X_s \in \mathbf{R}^{ds}$, we choose Λ large enough so that $B_\varepsilon(Y) \subset \Lambda$ for all $Y \in \Delta_m(X_s)$. This gives

$$\zeta^s \geq \frac{1}{m!} \int_{\Delta_m(X_s)} \zeta^{s+m} \exp\big(- \zeta |B_\varepsilon(X_{s+m})|\big)\, dX_{(s+1,s+m)} \, ,$$

and now $|B_\varepsilon(X_{s+m})| \leq \kappa_d \varepsilon^d (s+m)$ implies (5.1.2).

We finally fix an integer $K \geq 2$ and choose $s = 1$ in (5.1.6). Given $\varepsilon > 0$ and $x_1 \in \mathbf{R}^d$, we choose Λ large enough so that $B_\varepsilon(Y_K) \subset \Lambda$ for all $Y_K \in \Delta_K(x_1)$. This gives

$$\zeta - \zeta \exp\big(- \zeta |B_\varepsilon(x_1)|\big) \geq \sum_{k=2}^{K} \int_{\Delta_{k-1}(x_1)} \zeta^k \exp\big(- \zeta |B_\varepsilon(X_k)|\big)\, dX_{(2,k)} \, ,$$

and bounding the volumes of balls from above, we find

$$\zeta\big(1 - \exp(-\zeta\kappa_d\varepsilon^d)\big) \geq \sum_{k=1}^{K-1} \frac{\zeta^{k+1}}{k!} \exp\big(- \zeta\kappa_d (k+1)\varepsilon^d\big) \int_{\Delta_k(x_1)} dX_{(2,k+1)} \, .$$

It then suffices to let $K \to \infty$ to find (5.1.3). This ends the proof of Lemma 5.1.1. □

5.2. Functional spaces

To show the convergence of the series defining mild solutions (4.5.2) to the BBGKY hierarchy, we need to introduce some norms on the space of sequences $(\widetilde{f}^{(s)})_{s\geq 1}$. Given $\varepsilon \geq 0$, $\beta > 0$, an integer $s \geq 1$, and a function $g_s : \Omega_s \to \mathbf{R}$, we let

$$|g_s|_{\varepsilon,s,\beta} := \sup_{Z_s \in \Omega_s} \Big(|g_s(Z_s)| \exp\big(\beta E_\varepsilon(Z_s)\big)\Big) \tag{5.2.1}$$

where for $\varepsilon > 0$, the function E_ε is the s-particle Hamiltonian

$$E_\varepsilon(Z_s) := \sum_{1\le i\le s} \frac{|v_i|^2}{2} + \sum_{1\le i<k\le s} \Phi_\varepsilon(x_i - x_k)\,, \quad \text{with} \quad \Phi_\varepsilon(x) := \Phi\Big(\frac{x}{\varepsilon}\Big)\,, \tag{5.2.2}$$

and for $\varepsilon = 0$, E_0 is the free Hamiltonian:

$$E_0(Z_s) := \sum_{1\le i\le s} \frac{|v_i|^2}{2}\,. \tag{5.2.3}$$

Notation 5.2.1. — *For $\varepsilon \ge 0$ and $\beta > 0$, we denote $X_{\varepsilon,s,\beta}$ the Banach space of continuous functions $\Omega_s \to \mathbf{R}$ with finite $|\cdot|_{\varepsilon,s,\beta}$ norm.*

By Assumption 1.2.1, for $\varepsilon > 0$ (and $\beta > 0$) there holds $\exp(\beta E_\varepsilon(Z_s)) \to \infty$ as Z_s approaches $\partial\Omega_s$. This implies for $g_s \in X_{\varepsilon,s,\beta}$ the existence of an extension by continuity: $\bar g_s \in C^0(\mathbf{R}^{2ds};\mathbf{R})$ such that $\bar g_s \equiv 0$ on $\partial\Omega_s$, and $\bar g_s \equiv g$ on Ω_s.

For sequences of functions $G = (g_s)_{s\ge1}$, with $g_s : \Omega_s \to \mathbf{R}$, we let for $\varepsilon \ge 0$, $\beta > 0$, $\mu \in \mathbf{R}$,

$$\|G\|_{\varepsilon,\beta,\mu} := \sup_{s\ge1} \Big(|g_s|_{\varepsilon,s,\beta} \exp(\mu s) \Big)\,.$$

Notation 5.2.2. — *For $\varepsilon \ge 0$, $\beta > 0$, and $\mu \in \mathbf{R}$, we denote $\mathbf{X}_{\varepsilon,\beta,\mu}$ the Banach space of sequences $G = (g_s)_{s\ge1}$, with $g_s \in X_{\varepsilon,s,\beta}$ and $\|G\|_{\varepsilon,\beta,\mu} < \infty$.*

The following inclusions hold:

$$\text{if } \beta' \le \beta \text{ and } \mu' \le \mu, \text{ then} \quad X_{\varepsilon,s,\beta'} \subset X_{\varepsilon,s,\beta}\,, \quad \mathbf{X}_{\varepsilon,\beta',\mu'} \subset \mathbf{X}_{\varepsilon,\beta,\mu}\,. \tag{5.2.4}$$

Remark 5.2.3. — *These norms are classical in statistical physics, where probability measures are called "ensembles".*

At the canonical level, the ensemble $e^{-\beta E_\varepsilon(Z_s)} dZ_s$ is a normalization of the Lebesgue measure, where $\beta \sim T^{-1}$ (and T is the absolute temperature) specifies fluctuations of energy. The Boltzmann-Gibbs principle states that the average value of any quantity in the canonical ensemble is its equilibrium value at temperature T.

The micro-canonical level consists in restrictions of the ensemble to energy surfaces.

At the grand-canonical level the number of particles may vary, with variations indexed by chemical potential $\mu \in \mathbf{R}$.

5.3. Continuity estimates

We now establish bounds, in the above defined functional spaces, for the collision operators defined in (4.4.4)-(4.4.6), and for the total collision operator $\mathbf{C}_N$ (4.5.3). In $\mathcal{C}_{s,s+m}$, the sum in i over $[1, s]$ will imply a loss in μ, while the linear velocity factor will imply a loss in β. The losses are materialized in (5.3.2) by inequalities $\beta' < \beta$, $\mu' < \mu$.

Proposition 5.3.1. — *Given $\beta > 0$ and $\mu \in \mathbf{R}$, for $m \ge 1$ and $1 \le s \le N - m$, the collision operators $\mathcal{C}_{s,s+m}$ satisfy the bounds, for all $G_N = (g_s)_{1\le s\le N} \in \mathbf{X}_{\varepsilon,\beta,\mu}$,*

$$\big|\mathcal{C}_{s,s+m} g_{s+m}(Z_s)\big| \le \varepsilon^{m-1} C_d e^{m\kappa_d} (2\pi/\beta)^{md/2} \Big(s\beta^{-d/2} + \sum_{1\le i\le s} |v_i| \Big) e^{-\beta E_\varepsilon(Z_s)} |g_{s+m}|_{\varepsilon,s+m,\beta}\,, \tag{5.3.1}$$

for some $C_d > 0$ depending only on d.

If $\varepsilon < e^{\mu-\kappa_d}(\beta/2\pi)^{d/2}$, then for all $0 < \beta' < \beta$ and $\mu' < \mu$, the total collision operator $\mathbf{C}_N$ satisfies the bound

$$\|\mathbf{C}_N G_N\|_{\varepsilon,\beta',\mu'} \leq C_d(1+\beta^{-d/2})\Big(\frac{1}{\beta-\beta'}+\frac{1}{\mu-\mu'}\Big)\|G_N\|_{\varepsilon,\beta,\mu}\,, \tag{5.3.2}$$

for some $C_d > 0$ depending only on d.

Considering the case $m > 1$ in (5.3.1), for which the upper bound is $O(\varepsilon)$, we see that higher-order interactions are negligible in the Boltzmann-Grad limit.

Estimate (5.3.2), a continuity estimate with loss for the total collision operator $\mathbf{C}_N$, is not directly used in the following. In the existence proof (Chapter 6), we use instead the pointwise bound (5.3.1).

Proof. — We first consider the case $m \geq 2$. From the definition of $G^{(m-1)}_{\langle i,s+1\rangle}$ in (4.4.6), we obtain

$$\big|G^{(m-1)}_{\langle i,s+1\rangle}(g_{s+m})(Z_{s+1})\big| \leq |g_{s+m}|_{\varepsilon,s+m,\beta}\int_{\Delta_{m-1}(x_{s+1})\times\mathbf{R}^{d(m-1)}}\exp\big(-\beta E_\varepsilon(Z_{s+m})\big)dZ_{(s+2,s+m)}\,,$$

where the norm $|\cdot|_{\varepsilon,s,\beta}$ is defined in (5.2.1), and the Hamiltonian E_ε is defined in (5.2.2). For the collision operator defined in (4.4.4), this implies the bound

$$|\mathcal{C}_{s,s+m}g_{s+m}(Z_s)| \leq mC^m_{N-s}|g_{s+m}|_{\varepsilon,s+m,\beta}\times\sum_{1\leq i\leq s}I_{i,m}(V_s)\times J_{i,m}(X_s)\,, \tag{5.3.3}$$

where $I_{i,m}$ is the velocity integral

$$I_{i,m}(V_s) := \int_{\mathbf{R}^{dm}}\big(|v_{s+1}|+|v_i|\big)\exp\Big(-\frac{\beta}{2}\sum_{j=1}^{s+m}|v_j|^2\Big)dV_{(s+1,s+m)}\,,$$

and $J_{i,m}$ is the spatial integral

$$J_{i,m}(X_s) := \int_{S_\varepsilon(x_i)\times\Delta_{m-1}(x_{s+1})}\exp\Big(-\beta\sum_{1\leq j<k\leq s+m}\Phi_\varepsilon(x_j-x_k)\Big)d\sigma(x_{s+1})dX_{(s+2,s+m)}\,.$$

The velocity integral is a product of Gaussian integrals and can be exactly computed:

$$I_{i,m}(V_s) = (2\pi/\beta)^{(m-1)d/2}\Big((2\pi/\beta)^{d/2}|v_i|+(2/\beta)^d\Big)\exp\Big(-\frac{\beta}{2}\sum_{1\leq j\leq s}|v_j|^2\Big)\,. \tag{5.3.4}$$

For the spatial integral, there holds

$$\begin{aligned}J_{i,m}(X_s) &\leq \exp\Big(-\beta\sum_{1\leq j<k\leq s}\Phi_\varepsilon(x_j-x_k)\Big)|S_\varepsilon(x_i)|\times\sup_x\int_{\Delta_{m-1}(x)}dX_{(1,m-1)}\\ &\leq \exp\Big(-\beta\sum_{1\leq j<k\leq s}\Phi_\varepsilon(x_j-x_k)\Big)\times\kappa_d\varepsilon^{d-1}\times\Big((m-1)!\,\varepsilon^{(m-1)d}\exp(m\kappa_d)\Big)\,,\end{aligned}$$

where in the last bound we used identity (5.1.2) from Lemma 5.1.1 with $s = 1$ and $\zeta = \varepsilon^{-d}$. This implies

$$\begin{aligned}|\mathcal{C}_{s,s+m}g_{s+m}(Z_s)| \leq C_d\varepsilon^{m-1}\big((N-s)\varepsilon^{d-1}\big)^m e^{m\kappa_d}(2\pi/\beta)^{md/2}\Big(s\beta^{-d/2}+\sum_{1\leq i\leq s}|v_i|\Big)\\ \times e^{-\beta E_\varepsilon(Z_s)}|g_{s+m}|_{\varepsilon,s+m,\beta}\,.\end{aligned}$$

In the Boltzmann-Grad scaling $N\varepsilon^{d-1}\equiv 1$, this gives (5.3.1). Above and in the following, C_d denotes a positive constant which depends only on d, and which may change from line to line.

In the case $m = 1$, by definition of $G^{(0)}$ in (4.4.5), there holds

$$\left|G^{(0)}_{\langle i,s+1\rangle}(g_{s+1})(Z_{s+1})\right| \leq \exp\big(- \beta E_\varepsilon(Z_{s+1})\big)|g_{s+1}|_{\varepsilon,s+1,\beta}\,,$$

and this implies

$$|\mathcal{C}_{s,s+1}g_{s+1}(Z_s)| \leq (N-s)|g_{s+1}|_{\varepsilon,s+1,\beta} \exp\Big(- (\beta/2) \sum_{1\leq j<k\leq s} \Phi_\varepsilon(x_j - x_k)\Big) \times \sum_{1\leq i\leq s} I_{i,1}(V_s) \times \kappa_d \varepsilon^{d-1}\,,$$

from which (5.3.1) is deduced as above.

We turn to the proof of (5.3.2). From the pointwise inequality

$$\sum_{1\leq i\leq s} |v_i| \exp\Big(- (\gamma/2) \sum_{1\leq i\leq s} |v_i|^2\Big) \leq s^{1/2}(e\gamma)^{-1/2}\,, \qquad \gamma > 0\,,$$

we deduce for the above velocity integral $I_{i,m}(V_s)$ the bound, for $0 < \beta' < \beta$,

$$\sum_{1\leq i\leq s} \exp\Big((\beta'/2) \sum_{1\leq j\leq s} |v_j|^2\Big) I_{i,m}(V_s) \leq C_d(2\pi/\beta)^{md/2}\Big(s\beta^{-d/2} + s^{1/2}(\beta-\beta')^{-1/2}\Big)\,.$$

From the above bound in $J_{i,m}(X_s)$, we deduce immediately, for $0 < \beta' < \beta$,

$$\max_{1\leq i\leq s} \exp\Big(\beta' \sum_{1\leq j<k\leq s} \Phi_\varepsilon(x_j - x_k)\Big) J_{i,m}(X_s) \leq \kappa_d(m-1)! e^{m\kappa_d} \varepsilon^{md-1}\,.$$

With (5.3.3), these bounds yield, in the Boltzmann-Grad scaling,

$$\begin{aligned} e^{\beta' E_\varepsilon(Z_s)+\mu' s}\left|\mathcal{C}_{s,s+m}g_{s+m}(Z_s)\right| \leq \varepsilon^{m-1} C_d(2\pi/\beta)^{md/2} e^{m\kappa_d} e^{\mu' s}\big(s\beta^{-d/2} + s^{1/2}(\beta-\beta')^{-1/2}\big) \\ \times |g_{s+m}|_{\varepsilon,s+m,\beta}\,. \end{aligned}$$

Summing over m, we finally obtain, for $\mathbf{C}_N$ defined in (4.5.3),

$$\begin{aligned} \|\mathbf{C}_N G_N\|_{\varepsilon,\beta',\mu'} \leq C_d \|G_N\|_{\varepsilon,\beta,\mu} \sup_{1\leq s\leq N} \Big(\big(s\beta^{-d/2} + s^{1/2}(\beta-\beta')^{-1/2}\big) e^{-(\mu-\mu')s}\Big) \\ \times \sum_{1\leq m\leq N-s} e^{-m(\mu-\kappa_d)} \varepsilon^{m-1}(2\pi/\beta)^{md/2}\,. \end{aligned}$$

If ε is small enough so that $\varepsilon e^{\kappa_d-\mu}(2\pi/\beta)^{d/2} < 1$, then the above series is convergent, and

$$\sum_{1\leq m\leq N-s} e^{-m(\mu-\kappa_d)} \varepsilon^{m-1}(2\pi/\beta)^{md/2} \leq \frac{e^{\kappa_d-\mu}(2\pi/\beta)^{d/2}}{1 - \varepsilon e^{\kappa_d-\mu}(2\pi/\beta)^{d/2}}\,.$$

Finally,

$$\sup_{1\leq s\leq N} \Big(\big(s\beta^{-d/2} + s^{1/2}(\beta-\beta')^{-1/2}\big)e^{-(\mu-\mu')s}\Big) \leq e^{-1}(1+\beta^{-d/2})(\mu-\mu')^{-1} + (\beta-\beta')^{-1},$$

and this yields (5.3.2). Proposition 5.3.1 is proved. □

Remark 5.3.1. — *We do not use the extra decay provided by the contribution of the potential in the exponential of the Hamiltonian. This is quite obvious in the bound for $J_{i,m}(X_s)$ in the proof of Proposition* 5.3.1*, where we bound $e^{-\beta\sum_{1\leq j<k\leq s+m}\Phi_\varepsilon(x_j-x_k)}$ by $e^{-\beta\sum_{1\leq j<k\leq s}\Phi_\varepsilon(x_j-x_k)}$. Then, we might be tempted to replace E_ε by the free Hamiltonian E_0 in the definition of the functional spaces. The kinetic energy, however, is not a conserved quantity, so that in $X_{0,s,\beta}$ spaces the conservation of energy* (6.1.5) *does not hold.*

5.4. Continuity estimate for the limiting collision operator

Similarly to Chapter 2, we can define a limiting collision operator (see in particular (3.3.5) introduced in Chapter 3):
(5.4.1)

$$\mathcal{C}^0_{s,s+1} f^{(s+1)}(Z_s) := \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} \mathbb{1}_{\nu\cdot(v_{s+1}-v_i)>0}\, \nu\cdot(v_{s+1}-v_i) \\ \times \Big(f^{(s+1)}(t,x_1,v_1,\dots,x_i,v_i^*,\dots,x_s,v_s,x_i,v^*_{s+1}) - f^{(s+1)}(Z_s,x_i,v_{s+1})\Big) d\nu dv_{s+1}\,,$$

where v_i^* and v^*_{s+1} are obtained from v_i, v_{s+1} and ν by the inverse scattering operator σ_0^{-1} introduced in Chapter 3. The continuity estimate is as follows:

Proposition 5.4.1. — *Given $\beta>0$, $\mu\in\mathbf{R}$, the collision operator $\mathcal{C}^0_{s,s+1}$ satisfies the following bound, for all $g_{s+1}\in X_{0,s+1,\beta}$:*

$$\Big|\mathcal{C}^0_{s,s+1} g_{s+1}(Z_s)\Big| \leq C_d \beta^{-d/2}\Big(s\beta^{-d/2} + \sum_{1\leq i\leq s}|v_i|\Big) e^{-\beta E_0(Z_s)} |g_{s+1}|_{0,s+1,\beta}\,, \tag{5.4.2}$$

for some $C_d>0$ depending only on d.

Proof. — There holds

$$\big|\mathcal{C}^0_{s,s+1} g_{s+1}(Z_s)\big| \leq \sum_{1\leq i\leq s} \int_{\mathbf{S}^{d-1}\times\mathbf{R}^d} \big(|v_{s+1}|+|v_i|\big)\big(|g_{s+1}(v_i^*,v^*_{s+1})| + |g_{s+1}(v_i,v_{s+1})|\big) d\nu dv_{s+1},$$

omitting most of the arguments of g_{s+1} in the integrand. By definition of $|\cdot|_{0,s,\beta}$ norms and conservation of energy (3.1.4), there holds

$$|g_{s+1}(v_i^*,v^*_{s+1})| + |g_{s+1}(v_i,v_{s+1})| \leq \big(e^{-\beta E_0(Z_s^*)} + e^{-\beta E_0(Z_s)}\big)|g_{s+1}|_{0,\beta} \\ = 2e^{-\beta E_0(Z_s)}|g_{s+1}|_{0,s+1,\beta},$$

where Z_s^* is identical to Z_s except for v_i and v_{s+1} changed to v_i^* and v^*_{s+1}. This gives

$$\big|\mathcal{C}^0_{s,s+1} g_{s+1}(Z_s)\big| \leq C_d |g_{s+1}|_{0,s+1,\beta} e^{-\beta E_0(Z_s)} \sum_{1\leq i\leq s} I_{i,1}(V_s)\,,$$

borrowing notation from the proof of Proposition 5.3.1, and we conclude with (5.3.4). □

CHAPTER 6

LOCAL-IN-TIME WELL-POSEDNESS FOR THE BBGKY AND BOLTZMANN HIERARCHIES

We state and prove an existence and uniqueness result for mild solutions to the BBGKY hierarchy, defined in (4.5.4), which we reproduce here:

$$\widetilde{F}_N(t) = \mathbf{H}(t)\widetilde{F}_N(0) + \int_0^t \mathbf{H}(t-\tau)\mathbf{C}_N\widetilde{F}_N(\tau)\, d\tau\,, \qquad \widetilde{F}_N = (\widetilde{f}_N^{(s)})_{1\leq s\leq N}\,, \tag{6.0.1}$$

as well as for the limit Boltzmann hierarchy

$$F(t) = \mathbf{S}(t)F(0) + \int_0^t \mathbf{S}(t-\tau)\mathbf{C^0}F(\tau)\, d\tau\,, \qquad F = (f^{(s)})_{1\leq s}\,, \tag{6.0.2}$$

where the limiting collision operator $\mathbf{C^0}$ as well as the free-particle flow $\mathbf{S}(t)$ are defined in (4.5.5).

6.1. Functional spaces and statement of the result

Existence and uniqueness for (6.0.1) will take place in spaces of $\mathbf{X}_{\varepsilon,\beta,\mu}$-valued functions of time (see Notation 5.2.2 page 34), where the indices β and μ themselves depend on time: in the sequel we choose for simplicity a linear dependence in time, though other, decreasing functions of time could be chosen just as well. Such a time dependence on the parameters of the function spaces is a situation which occurs whenever continuity estimates involve a loss, which is the case here since the continuity estimates on the collision operators lead to a deterioration in the parameters β and μ. We refer to Section 6.3 for some comments.

Definition 6.1.1. — *Given $T > 0$, a positive function $\boldsymbol{\beta}$ and a real valued function $\boldsymbol{\mu}$ defined on $[0,T]$ we denote $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ the space of functions $G : t \in [0,T] \mapsto G(t) = (g_s(t))_{1\leq s} \in \mathbf{X}_{\varepsilon,\beta(t),\mu(t)}$, such that for all $Z_s \in \mathbf{R}^{2ds}$, the map $t \in [0,T] \mapsto g_s(t, Z_s)$ is measurable, and*

$$|\!|\!| G |\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} := \sup_{0\leq t\leq T} \|G(t)\|_{\varepsilon,\beta(t),\mu(t)} < \infty\,. \tag{6.1.3}$$

We define similarly

$$|\!|\!| G |\!|\!|_{0,\boldsymbol{\beta},\boldsymbol{\mu}} := \sup_{0\leq t\leq T} \|G(t)\|_{0,\beta(t),\mu(t)}\,.$$

The existence result for the BBGKY hierarchy we shall prove is the following.

***Theorem 2* (Uniform existence for the BBGKY hiearchy).** — *Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. There are $T > 0$ and $\lambda > 0$ such that $\boldsymbol{\beta} := \beta_0 - \lambda T > 0$, as well as $\varepsilon_0 > 0$ and $C > 0$, such that for all $0 < \varepsilon \leq \varepsilon_0$, defining $\boldsymbol{\mu} := \mu_0 - \lambda T$, any family of initial marginals $\widetilde{F}_N(0) = \big(\widetilde{f}_N^{(s)}(0)\big)_{s\in\mathbf{N}^*}$ in $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ gives rise to a unique solution $\widetilde{F}_N(t) = (\widetilde{f}_N^{(s)}(t))_{1\leq s\leq N}$ in $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ to the BBGKY hierarchy* (6.0.1) *in the Boltzmann-Grad scaling $N\varepsilon^{d-1} = 1$. It satisfies the following bound:*

$$|\!|\!| \widetilde{F}_N |\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \leq C \|\widetilde{F}_N(0)\|_{\varepsilon,\beta_0,\mu_0} \, .$$

This is a *uniform* existence result, in the sense that the existence time T does not depend on the number of particles N, which of course is crucial in the perspective of the limit $N \to \infty$. Note that actually the only assumption made is on *bounds* on the initial family of marginals.

For fixed $\varepsilon > 0$, the uniqueness statement in Theorem 2 allows to define a maximal existence time $T_*(\varepsilon)$. However we expect $\sup\limits_{\varepsilon>0} T_*(\varepsilon)$ to be attained at $\varepsilon = 0$, which precludes the definition of a maximal existence time for the ε-dependent family of hierarchies. We can however give a uniform bound from below for an existence time in Theorem 2: the following result is a corollary of the proof of Theorem 2, its proof is provided at the end of Section 6.2.

Corollary 6.1.2. — *Given $\beta_0 > 0$, $\mu_0 \in \mathbf{R}$, for some constant $C_d > 0$, given*

$$T := C_d e^{\mu_0} (1 + 2\beta_0^{d/2})^{-1} \max_{\beta\in[0,\beta_0]} \beta e^{-\beta} (\beta_0 - \beta)^d \tag{6.1.4}$$

then for all $0 < \varepsilon \leq \varepsilon_0$, the solution to the BBGKY hierarchy with data $\widetilde{F}_N(0) = \big(\widetilde{f}_N^{(s)}(0)\big)_{s\in\mathbf{N}^*}$ belonging to $\mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ is defined on $[0, T]$.

Remark 6.1.3. — *For $d \ll \beta_0$, there holds $\max\limits_{\beta\in[0,\beta_0]} \beta e^{-\beta} (\beta_0 - \beta)^d = \beta_0^d \big(1 + o(1)\big)$, hence an existence time of the order of $e^{\mu_0}\beta_0^{d/2}$.*

A similar existence result as Theorem 2 can be obtained for the Boltzmann hiearchy.

***Theorem 3* (Existence for the Boltzmann hiearchy).** — *Let $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$ be given. Then with the same notation as Theorem 2, any family of initial marginals $F(0) = \big(f^{(s)}(0)\big)_{1\leq s} \in \mathbf{X}_{0,\beta_0,\mu_0}$ gives rise to a unique solution $F(t) = (f^{(s)}(t))_{1\leq s}$ in $\mathbf{X}_{0,\boldsymbol{\beta},\boldsymbol{\mu}}$ to the Boltzmann hierarchy* (6.0.2)*. It satisfies the following bound:*

$$|\!|\!| F |\!|\!|_{0,\boldsymbol{\beta},\boldsymbol{\mu}} \leq C \|F(0)\|_{0,\beta_0,\mu_0} \, .$$

The proof of Theorems 2 and 3 is typical of analytical-type results, such as the classical Cauchy-Kowalevskaya theorem. We follow here Ukai's approach [**40**], which turns out to be remarkably short and self-contained; the different approach of Nirenberg [**32**] and Nishida [**33**] would allow for direct use of the loss estimate (5.3.2). Let us give the main steps of the proof: we start by noting that the conservation of energy for the s-particle flow is reflected in identities

$$\begin{aligned} |\mathbf{H}_s(t) g_s|_{\varepsilon,s,\beta} &= |g_s|_{\varepsilon,s,\beta} \quad \text{and} \quad \|\mathbf{H}(t) G_N\|_{\varepsilon,\beta,\mu} = \|G_N\|_{\varepsilon,\beta,\mu} \, , \\ |\mathbf{S}_s(t) g_{0,s}|_{0,s,\beta} &= |g_s|_{0,s,\beta} \quad \text{and} \quad \|\mathbf{S}(t) G_0\|_{0,\beta,\mu} = \|G_0\|_{0,\beta,\mu} \, , \end{aligned} \tag{6.1.5}$$

for all $\beta > 0$, $\mu \in \mathbf{R}$, $g_s \in X_{\varepsilon,s,\beta}$, $g_{0,s} \in X_{0,s,\beta}$, $G_N = (g_s)_{1\leq s\leq N} \in \mathbf{X}_{\varepsilon,\beta,\mu}$, $G_0 = (g_{0,s})_{1\leq s} \in \mathbf{X}_{0,\beta,\mu}$, and all $t \geq 0$. Next assume that there is a constant $c < 1$ such that for ε_0 small enough (depending on $c, \beta_0, \mu_0, \lambda$ and T) there holds the bound

$$\forall\, 0 < \varepsilon \leq \varepsilon_0 \, , \quad \Big|\!\Big|\!\Big| \int_0^t \mathbf{H}(t - t') \mathbf{C}_N G_N(t')\, dt' \Big|\!\Big|\!\Big|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \leq c \, |\!|\!| G_N |\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \, . \tag{6.1.6}$$

This estimate is the object of Lemma 6.2.2 below. Under (6.1.6), the linear operator

$$\mathfrak{L}: \quad G_N \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} \mapsto \Big(t \mapsto G_N(t) - \int_0^t \mathbf{H}(t-t')\mathbf{C}_N G_N(t')\,dt' \Big) \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$$

is is linear continuous from $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ to itself with norm strictly smaller than one. In particular, it is invertible in the Banach algebra $\mathcal{L}(\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}})$. Next given $\widetilde{F}_N(0) \in \mathbf{X}_{\varepsilon,\beta_0,\mu_0}$, by conservation of energy (6.1.5), inclusions (5.2.4) and decay of $t \mapsto \beta_0 - \lambda t$ and $t \mapsto \mu_0 - \lambda t$, there holds

$$\big(t \mapsto \mathbf{H}(t)\widetilde{F}_N(0) \big) \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\,.$$

Hence, there exists a unique solution $\widetilde{F}_N \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ to $\mathfrak{L}\widetilde{F}_N = \mathbf{H}(\cdot)\widetilde{F}_N(0)$, an equation which is equivalent to (6.0.1).

The reasoning is identical for Theorem 3, replacing (6.1.6) by

$$\Big|\Big|\Big| \int_0^t \mathbf{S}(t-t')\mathbf{C}^0 G(t')\,dt' \Big|\Big|\Big|_{0,\boldsymbol{\beta},\boldsymbol{\mu}} \leq c\, |||G|||_{0,\boldsymbol{\beta},\boldsymbol{\mu}}\,. \tag{6.1.7}$$

The next section is devoted to the proof of (6.1.6) and (6.1.7).

6.2. Continuity estimates

As explained in the previous paragraph, we need to prove (6.1.6), and its counterpart (6.1.7) for the Boltzmann operators. Let us first prove a continuity estimate based on Proposition 5.3.1.

Lemma 6.2.1. — *Under the assumptions of Theorem* 2, *there holds the bound, for* $0 \leq t \leq T$,

$$e^{s(\mu_0-\lambda t)}\Big| \int_0^t \mathbf{H}_s(t-t')\mathcal{C}_{s,s+1}g_{s+1}(t')\,dt' \Big|_{\varepsilon,s,\beta_0-\lambda t} \leq \bar{c}(\beta_0,\mu_0,\lambda,T)|||G_N|||_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\,, \tag{6.2.8}$$

for all $G_N = (g_{s+1})_{1\leq s\leq N} \in \mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$, *with* $\bar{c}(\beta_0,\mu_0,\lambda,T)$ *computed explicitly in* (6.2.14) *below.*

Proof. — Let us define

$$\beta_0^\lambda(t) := \beta_0 - \lambda t \quad \text{and} \quad \mu_0^\lambda(t) := \mu_0 - \lambda t\,, \tag{6.2.9}$$

so that $\boldsymbol{\beta} = \beta_0^\lambda(T)$ and $\boldsymbol{\mu} = \mu_0^\lambda(T)$. By conservation of energy (6.1.5), there holds the bound

$$\Big| \int_0^t \mathbf{H}(t-t')\mathcal{C}_{s,s+1}g_{s+1}(t')\,dt' \Big|_{\varepsilon,s,\beta_0^\lambda(t)} \leq \sup_{Z_s\in\mathbf{R}^{2ds}} \int_0^t e^{\beta_0^\lambda(t)E_\varepsilon(Z_s)}\big|\mathcal{C}_{s,s+1}g_{s+1}(t',Z_s)\big|\,dt'\,.$$

Estimate (5.3.1) from Proposition 5.3.1 gives

$$\begin{aligned} &e^{\beta_0^\lambda(t)E_\varepsilon(Z_s)}\big|\mathcal{C}_{s,s+1}g_{s+1}(t',Z_s)\big| \\ &\leq C_d e^{\kappa_d}(2\pi/\beta_0^\lambda(t'))^{d/2}|g_{s+1}(t')|_{\varepsilon,s+1,\beta_0^\lambda(t')}\Big(s(\beta_0^\lambda(t'))^{-d/2} + \sum_{1\leq i\leq s}|v_i|\Big)e^{\lambda(t'-t)E_\varepsilon(Z_s)}\,. \end{aligned}$$

By definition of norms $\|\cdot\|_{\varepsilon,\beta,\mu}$ and $|||\cdot|||_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ we have

$$\begin{aligned} |g_{s+1}(t')|_{\varepsilon,s+1,\beta_0^\lambda(t')} &\leq e^{-(s+1)\mu_0^\lambda(t')}\|G_N(t')\|_{\varepsilon,\beta_0^\lambda(t'),\mu_0^\lambda(t')} \\ &\leq e^{-(s+1)\mu_0^\lambda(t')}|||G_N|||_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\,. \end{aligned} \tag{6.2.10}$$

The above bounds yield, since β_0^λ and μ_0^λ are nonincreasing,

$$
\begin{aligned}
&e^{s\mu_0^\lambda(t)}\Big|\int_0^t \mathbf{H}(t-t')\mathcal{C}_{s,s+1}g_{s+1}(t')\,dt'\Big|_{\varepsilon,s,\beta_0^\lambda(t)} \\
&\qquad\leq |\!|\!| G_N |\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}} C_d e^{\kappa_d-\mu_0^\lambda(T)}(2\pi/\beta_0^\lambda(T))^{d/2}\sup_{Z_s\in\mathbf{R}^{2ds}}\int_0^t \overline{C}(t',t,Z_s)\,dt'\,,
\end{aligned}
\tag{6.2.11}
$$

where, for $t'\leq t$,

$$
\overline{C}(t',t,Z_s) := \Big(s(\beta_0^\lambda(t'))^{-d/2}+\sum_{1\leq i\leq s}|v_i|\Big)e^{\lambda(t'-t)(s+E_\varepsilon(Z_s))}\,. \tag{6.2.12}
$$

Since

$$
\sup_{Z_s\in\mathbf{R}^{2ds}}\int_0^t\overline{C}(t',t,Z_s)\,dt'\leq\frac{C_d}{\lambda}\Big(1+\frac{1}{(\beta_0^\lambda(T))^{d/2}}\Big)\,, \tag{6.2.13}
$$

there holds finally

$$
e^{s\mu_0^\lambda(t)}\Big|\int_0^t \mathbf{H}(t-t')\mathcal{C}_{s,s+1}g_{s+1}(t')\,dt'\Big|_{\varepsilon,s,\beta_0^\lambda(t)}\leq\bar{c}(\beta_0,\mu_0,\lambda,T)|\!|\!| G_N |\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\,,
$$

where, with a possible change of the constant C_d,

$$
\bar{c}(\beta_0,\mu_0,\lambda,T):=C_d e^{-\mu_0^\lambda(T)}\frac{1}{\lambda(\beta_0^\lambda(T))^{d/2}}\Big(1+\frac{1}{(\beta_0^\lambda(T))^{d/2}}\Big)\,. \tag{6.2.14}
$$

The result follows. □

In the next lemma, the definition (6.2.14) of $\bar{c}$ provides directly (6.1.6).

Lemma 6.2.2. — *Under the assumptions of Theorem 2, and for ε_0 small enough (depending on β_0,μ_0,λ and T) there holds the bound, for $0\leq t\leq T$,*

$$
\forall\, 0<\varepsilon\leq\varepsilon_0\,,\quad \Big\|\int_0^t\mathbf{H}(t-t')\mathbf{C}_N G_N(t')\,dt'\Big\|_{\varepsilon,\beta_0-\lambda t,\mu_0-\lambda t}\leq 2\bar{c}(\beta_0,\mu_0,\lambda,T)|\!|\!| G_N |\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\,, \tag{6.2.15}
$$

for all $G_N=(g_s)_{1\leq s\leq N}\in\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}},$, where $\bar{c}$ is defined in (6.2.14).

Proof. — We follow closely the proof of Lemma 6.2.1. The difference is that here we take into account higher-order collision operators $\mathcal{C}_{s,s+m}$, with $m\geq 2$.

Using notation (6.2.9), Estimate (5.3.1) from Proposition 5.3.1 gives

$$
\begin{aligned}
&e^{\beta_0^\lambda(t)E_\varepsilon(Z_s)}\big|\mathcal{C}_{s,s+m}g_{s+m}(t',Z_s)\big| \\
&\leq\varepsilon^{m-1}C_d e^{m\kappa_d}(2\pi/\beta_0^\lambda(t'))^{md/2}|g_{s+m}(t')|_{\varepsilon,s+m,\beta_0^\lambda(t')}\Big(s(\beta_0^\lambda(t'))^{-d/2}+\sum_{1\leq i\leq s}|v_i|\Big)e^{\lambda(t'-t)E_\varepsilon(Z_s)}\,.
\end{aligned}
$$

Using also (6.2.10) with $s+1$ replaced by $s+m$, we get

$$
\begin{aligned}
&\Big\|\int_0^t\mathbf{H}(t-t')\mathbf{C}_N G_N(t')\,dt'\Big\|_{\varepsilon,\beta_0^\lambda(t),\mu_0^\lambda(t)} \\
&\qquad\leq|\!|\!| G_N |\!|\!|_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}\Big(\sum_{1\leq m\leq N-s}C_m\Big)\sup_{Z_s\in\mathbf{R}^{2ds}}\int_0^t\overline{C}(t,t',Z_s)\,dt'\,,
\end{aligned}
\tag{6.2.16}
$$

where $C_m:=C_d\varepsilon^{m-1}e^{m(\kappa_d-\mu_0^\lambda(T))}(2\pi/\beta_0^\lambda(T))^{md/2}$, and $\overline{C}$ is defined in (6.2.12) and satisfies (6.2.13). Under the assumption that

$$
\varepsilon_0 e^{\kappa_d-\mu_0^\lambda(T)}(2\pi/\beta_0^\lambda(T))^{d/2}<1/2\,, \tag{6.2.17}
$$

we find

$$\sum_{1\leq m\leq N-s} C_m \leq 2C_d e^{-\mu_0^\lambda(T)}(\beta_0^\lambda(T))^{-d/2}\,. \tag{6.2.18}$$

The upper bounds in (6.2.13) and (6.2.18) are independent of s, and their product is equal to $2\bar{c}(\beta_0,\mu_0,\lambda,T)$. Taking the supremum in s in (6.2.16) then yields (6.2.15). □

The proof of the corresponding result (6.1.7) for the Boltzmann hiearchy is identical to the first part of the proof of Proposition 8.1.1, since the estimates for $\mathcal{C}^0_{s,s+1}$ and $\mathcal{C}_{s,s+1}$ are essentially identical (compare estimate (5.3.1) from Proposition 5.3.1 with estimate (5.4.2) from Proposition 5.4.1).

Proof of Corollary 6.1.2. — Given T, we are looking for $\lambda>0$ and $\varepsilon_0>0$ such that (6.2.17) holds and, say

$$C_d\big(2+(\beta_0-\lambda T)^{-d/2}\big)e^{-\mu_0+\lambda T}(\beta_0-\lambda T)^{-d/2}=\frac{\lambda}{3}\cdot \tag{6.2.19}$$

Indeed, if such a λ exists, we can then define $\varepsilon_0=(1/3)e^{-\kappa_d+\mu_0-\lambda T}(2\pi/(\beta_0-\lambda T))^{-d/2}$, and (6.2.17) holds. With $\beta=\lambda T\in(0,\beta_0)$, condition (6.2.19) becomes

$$\begin{aligned}T&=C_d e^{\mu_0}\beta e^{-\beta}\frac{(\beta_0-\beta)^d}{1+2(\beta_0-\beta)^{d/2}}\\&\geq C_d e^{\mu_0}(1+2\beta_0^{d/2})^{-1}\beta e^{-\beta}(\beta_0-\beta)^d\,,\end{aligned}$$

and (6.1.4) follows. □

In particular, given an existence time T for the BBGKY hierarchy, in the sense of Theorem 2, then T is an existence time for the Boltzmann hierarchy (6.0.2).

6.3. Some remarks on the strategy of proof

The key in the proof of (6.1.6) is *not* to apply Minkowski's integral inequality, which would indeed lead here to

$$\Big\|\int_0^t \mathbf{H}(t-t')\mathbf{C}_N G_N(t')\,dt'\Big\|_{\varepsilon,\beta_0^\lambda(t),\mu_0^\lambda(t)}\leq\int_0^t\big\|\mathbf{C}_N G_N(t')\big\|_{\varepsilon,\beta_0^\lambda(t),\mu_0^\lambda(t)}\,dt'\,,$$

by (6.1.5), and then to a divergent integral of the type

$$\Big\|\int_0^t \mathbf{H}(t-t')\mathbf{C}_N G_N(t')\,dt'\Big\|_{\varepsilon,\beta_0^\lambda(t),\mu_0^\lambda(t)}\leq C\big(\beta_0^\lambda(T),\mu_0^\lambda(T)\big)\int_0^t\Big(\frac{1}{\beta_0^\lambda(t')-\beta_0^\lambda(t)}+\frac{1}{\mu_0^\lambda(t')-\mu_0^\lambda(t)}\Big)\,dt'\,.$$

The difference is that by Minkowski the upper bound appears as the time integral of a supremum in s, while in the proof of Lemma 6.2.2, and hence of (6.1.6), the upper bound is a supremum in s of a time integral.

As pointed out in Section 6.1, other proofs of Theorems 2 and 3 can be devised, using tools inspired by the proof of the Cauchy-Kowalevskaya theorem: we recall for instance the approaches of [**32**] and [**33**], as well as [**31**] and [**27**].

CHAPTER 7

ADMISSIBLE INITIAL DATA AND MAIN RESULT

We state here our main result, describing convergence of mild solutions to the BBGKY hierarchy (6.0.1) to mild solutions of the Boltzmann hierarchy (6.0.2). This result implies in particular Theorem 1 stated in the Introduction. Existence and uniqueness results for both hierarchies were previously given in Chapter 5, as Theorem 2 page 40 and Theorem 3 page 40.

The first part of this chapter is devoted to a precise description of Boltzmann initial data which are *admissible*, i.e., which give rise to solutions for which the convergence result holds. This involves discussing the notion of "quasi-independence" mentioned in the Introduction, via a conditioning of the initial data. Then we state Theorem 4 and sketch the main steps of its proof.

7.1. Quasi-independence

In this paragraph we discuss the notion of "quasi-independent" initial data. We first define admissible Boltzmann initial data, meaning data which can be attained from BBGKY initial data (meaning bounded families of truncated marginals) by a limiting procedure, and then show how to "condition" the initial BBGKY initial data so as to converge towards admissible Boltzmann initial data. Finally we characterize admissible Boltzmann initial data.

7.1.1. Admissible Boltzmann data. — Let us define admissible Boltzmann initial data.

***Definition 7.1.1* (Admissible Boltzmann data).** — *Admissible Boltzmann data are defined as families $F_0 = (f_0^{(s)})_{s\geq 1}$, with each $f_0^{(s)}$ nonnegative, integrable and continuous over Ω_s, such that*

$$\int_{\mathbf{R}^{2d}} f_0^{(s+1)}(Z_s, z_{s+1})\, dz_{s+1} = f_0^{(s)}(Z_s)\,, \tag{7.1.1}$$

and which are limits of BBGKY initial data $\widetilde{F}_{0,N} = (\widetilde{f}_{0,N}^{(s)})_{1\leq s\leq N} \in \mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ in the following sense: it is assumed that

$$\sup_{N\geq 1} \|\widetilde{F}_{0,N}\|_{\varepsilon,\beta_0,\mu_0} < \infty\,, \quad \textit{for some } \beta_0 > 0\,,\ \mu_0 \in \mathbf{R}\,,\ \textit{as } N\varepsilon^{d-1} \equiv 1\,, \tag{7.1.2}$$

$$\textit{and} \quad \widetilde{f}_{0,N}^{(s)}(Z_s) = \int_{\mathbf{R}^{2d(N-s)}} 1\!\!1_{\mathcal{D}_N^s}(X_N)\widetilde{f}_{0,N}^{(N)}(Z_N)dZ_{(s+1,N)}\,, \quad 1 \leq s < N\,, \tag{7.1.3}$$

and that the following convergence holds:

$$\widetilde{f}^{(s)}_{0,N} \longrightarrow f_0^{(s)}\,, \quad \textit{for each } s\,, \textit{ as } N \to \infty \textit{ with } N\varepsilon^{d-1} \equiv 1\,, \textit{ locally uniformly in } \Omega_s\,. \tag{7.1.4}$$

In this section we shall prove the following result.

Proposition 7.1.1. — *The set of admissible Boltzmann data, in the sense of Definition* 7.1.1*, is the set of families of marginals* F_0 *as in* (7.1.1) *satisfying a uniform bound* $\|F_0\|_{0,\beta_0,\mu_0} < \infty$.

7.1.2. Conditioning. — We first consider "chaotic" configurations, corresponding to tensorized initial measures, or initial densities which are products of one-particle distributions:

$$f_0^{\otimes s}(Z_s) = \prod_{1 \le i \le s} f_0(z_i)\,, \qquad 1 \le s \le N\,, \tag{7.1.5}$$

where f_0 is nonnegative, normalized, and belongs to some $X_{0,1,\beta_0}$ space (see Definition 5.2.1 page 34):

$$f_0 \ge 0\,, \quad \int_{\mathbf{R}^{2d}} f_0(z)dz = 1\,, \quad f_0 \in X_{0,1,\beta_0} \quad \text{for some } \beta_0 > 0\,. \tag{7.1.6}$$

An important observation is that for $(f_0^{\otimes s})_{1\le s\le N}$ defined by (7.1.5), with f_0 satisfying (7.1.6), there holds in general $\sup\limits_{N\ge 1} \|(f_0^{\otimes s})_{1\le s\le N}\|_{\varepsilon,\beta,\mu} = +\infty$, for all $\beta > 0$, $\mu \in \mathbf{R}$. Indeed, the correction in the Hamiltonian due to the potential Φ_ε produces errors of size $O(1)$ in s-particle configuration subdomains such that $|x_i - x_j| \le \varepsilon$. These subdomains are *not* asymptotically small, even in the thermodynamical limit $N\varepsilon^{d-1} \equiv 1$.

This calls for cancelling out the contribution of the potential, by consideration of

$$f_{0,N}^{\otimes N}(Z_N) := \exp\Big(- \beta_0 \sum_{1\le i<j\le N} \Phi_\varepsilon(x_i - x_j)\Big) \prod_{1\le i\le N} f_0(z_i)\,, \quad Z_N \in \Omega_N\,. \tag{7.1.7}$$

With this definition, there holds the identity

$$|f_{0,N}^{\otimes N}|_{\varepsilon,N,\beta_0} = |f_0|^N_{0,N,\beta_0}\,, \tag{7.1.8}$$

where the norms $|\cdot|_{\varepsilon,N,\beta_0}$ and $|\cdot|_{0,N,\beta_0}$ are defined page 33. Indeed, using the notation for the Hamiltonian introduced in Section 5.2 page 33, there holds

$$|f_{0,N}^{\otimes N}|_{\varepsilon,N,\beta_0} = \sup_{Z_s \in \mathbf{R}^{2ds}} e^{\beta_0 E_\varepsilon(Z_N)} e^{-\beta_0 E_\varepsilon(X_N,0)} \prod_{1\le s\le N} |f_0(z_i)| = \sup_{Z_N \in \mathbf{R}^{2dN}} \prod_{1\le i\le N} e^{\beta_0 |v_i|^2/2} |f_0(z_i)|\,,$$

and the last term in the right-hand side above is equal to $|f_0|^N_{0,N,\beta_0}$.

The property of normalization is then preserved by introduction of the partition function

$$\mathcal{Z}_N := \int_{\mathbf{R}^{2dN}} f_{0,N}^{\otimes N}(Z_N)\, dZ_N\,, \quad 1 \le s \le N\,, \tag{7.1.9}$$

and the definition of *conditioned datum built on* f_0 as $\mathcal{Z}_N^{-1} f_{0,N}^{\otimes N}$. This operation is called *conditioning on energy surfaces*, and is a classical tool in statistical mechanics (see [**18, 29, 30**] for instance).

The partition function defined in (7.1.9) satisfies the next result, which will be useful in the following.

Lemma 7.1.2. — *Given* f_0 *satisfying* (7.1.6)*, there holds for* $1 \le s \le N$ *the bound*

$$1 \le \mathcal{Z}_N^{-1} \mathcal{Z}_{N-s} \le \big(1 - \varepsilon\kappa_d |f_0|_{L^\infty L^1}\big)^{-s},$$

in the scaling $N\varepsilon^{d-1} \equiv 1$*, where* $|f_0|_{L^\infty L^1}$ *denotes the* $L^\infty(\mathbf{R}^d_x, L^1(\mathbf{R}^d_v))$ *norm of* f_0*, and* κ_d *denotes the volume of the unit ball in* $\mathbf{R}^d$.

Proof. — From the trivial lower bound

$$\exp\left(-\beta_0 \Phi_\varepsilon(x_i - x_j)\right) \geq 1\!\!1_{|x_i - x_j| > \varepsilon} ,$$

we deduce the lower bound

$$\mathcal{Z}_{s+1} \geq \int_{\mathbf{R}^{2d(s+1)}} \exp\Big(- \beta_0 E_\varepsilon(X_s, 0)\Big)\Big(\prod_{i=1}^{s} 1\!\!1_{|x_i - x_{s+1}| > \varepsilon}\Big) f_0^{\otimes(s+1)}(Z_{s+1})\, dZ_{(1,s+1)} ,$$

with $E_\varepsilon(X_s, 0) = \displaystyle\sum_{1 \leq i < j \leq s} \Phi_\varepsilon(x_i - x_j)$, in accordance with the definition of the Hamiltonian (5.2.2). By Fubini, we have

$$\begin{aligned}\int_{\mathbf{R}^{2d(s+1)}} &\exp\Big(- \beta_0 E_\varepsilon(X_s, 0)\Big)\Big(\prod_{1 \leq i \leq s} 1\!\!1_{|x_i - x_{s+1}| > \varepsilon}\Big) f_0^{\otimes(s+1)}(Z_{s+1})\, dZ_{(1,s+1)} \\ &= \int_{\mathbf{R}^{2ds}} \left(\int_{\mathbf{R}^{2d}} \Big(\prod_{1 \leq i \leq s} 1\!\!1_{|x_i - x_{s+1}| > \varepsilon}\Big) f_0(z_{s+1}) dz_{s+1} \right) f_{0,N}^{\otimes s}(Z_s) dZ_{(1,s)} .\end{aligned}$$

Since

$$\int_{\mathbf{R}^{2d}} \Big(\prod_{1 \leq i \leq s} 1\!\!1_{|x_i - x_{s+1}| > \varepsilon}\Big) f_0(z_{s+1}) dz_{s+1} \geq |f_0|_{L^1} - \kappa_d s \varepsilon^d |f_0|_{L^\infty L^1} ,$$

we deduce from the above, by nonnegativity of $f_{0,N}^{\otimes s}$, the lower bound

$$\mathcal{Z}_{s+1} \geq \mathcal{Z}_s \big(|f_0|_{L^1} - \kappa_d s \varepsilon^d |f_0|_{L^\infty L^1} \big) ,$$

implying by induction

$$\mathcal{Z}_N \geq \mathcal{Z}_{N-s} \prod_{j=N-s}^{N-1} (1 - j\varepsilon^d \kappa_d |f_0|_{L^\infty L^1}) \geq \mathcal{Z}_{N-s} \big(1 - \varepsilon \kappa_d |f_0|_{L^\infty L^1}\big)^s ,$$

where we used $s \leq N$ and the scaling $N\varepsilon^{d-1} \equiv 1$. That proves the lemma. □

7.1.3. Characterization of admissible Boltzmann initial data. — The aim of this paragaph is to prove Proposition 7.1.1.

Let us start by proving the following statement, which provides examples of admissible Boltzmann initial data, in terms of tensor products.

Proposition 7.1.2. — *Given f_0 satisfying* (7.1.6)*, define $f_{0,N}^{\otimes N}$ as in* (7.1.7)*, and let $f_{0,N}^{(N)}$ be a conditioned datum built on f_0 :*

$$f_{0,N}^{(N)} := \mathcal{Z}_N^{-1} f_{0,N}^{\otimes N} . \tag{7.1.10}$$

Then, families $(\widetilde{f}_{0,N}^{(s)})_{1 \leq s \leq N}$ of truncated marginals of $f_{0,N}^{(N)}$, as defined in (7.1.3)*, satisfy* (7.1.2) *for any μ_0 such that $e^{\mu_0} |f_0|_{0,\beta_0} < 1$. Moreover the data $F_0 = (f_0^{\otimes s})_{s \geq 1}$ is admissible Boltzmann initial data associated with $\widetilde{F}_{0,N} = (\widetilde{f}_{0,N}^{(s)})_{1 \leq s \leq N}$.*

Proof. — In a first step, we prove that untruncated marginals $F_{0,N} := \big(f_{0,N}^{(s)}\big)_{s \leq N}$ satisfy uniform bounds. In a second step, we prove that untruncated marginals converge uniformly in Ω_s towards $f_{0,N}^{\otimes s}$. We finally prove that truncated marginals converge as well.

First step. The trivial bound

$$\exp\Big(-\beta_0 \sum_{i\le s,s+1\le j} \Phi_\varepsilon(x_i-x_j)\Big)\le 1$$

yields, using notation (7.1.7),

$$f_{0,N}^{(s)}(Z_s)\le \mathcal{Z}_N^{-1} f_{0,N}^{\otimes s}(Z_s)\int_{\mathbf{R}^{2d(N-s)}}\exp\Big(-\beta_0\sum_{s+1\le i<j\le N}\Phi_\varepsilon(x_i-x_j)\Big)\prod_{s+1\le i\le N} f_0(z_i)\,dZ_{(s+1,N)}\,.$$

By symmetry,

$$\int_{\mathbf{R}^{2d(N-s)}}\exp\Big(-\beta_0\sum_{s+1\le i<j\le N}\Phi_\varepsilon(x_i-x_j)\Big)\prod_{s+1\le i\le N} f_0(z_i)\,dZ_{(s+1,N)}=\mathcal{Z}_{N-s}\,, \tag{7.1.11}$$

and this gives

$$\begin{aligned} f_{0,N}^{(s)} &\le \mathcal{Z}_N^{-1}\mathcal{Z}_{N-s} f_{0,N}^{\otimes s}\\ &\le \big(1-\varepsilon\kappa_d|f_0|_{L^\infty L^1}\big)^{-s} f_{0,N}^{\otimes s}\,,\end{aligned}$$

the second inequality by Lemma 7.1.2.

By $2x+\ln(1-x)\ge 0$ for $x\in[0,1/2]$, there holds

$$(1-\varepsilon\kappa_d|f_0|_{L^\infty L^1})^{-s}\le e^{2s\varepsilon\kappa_d|f_0|_{L^\infty L^1}}\,,\qquad \text{if } 2\varepsilon\kappa_d|f_0|_{L^\infty L^1}<1\,, \tag{7.1.12}$$

so that for N larger than some N_0 (equivalently, for ε small enough),

$$e^{s\mu_0}\big|f_{0,N}^{(s)}\big|_{\varepsilon,\beta_0}\le e^{s(\mu_0+2\varepsilon\kappa_d|f_0|_{L^\infty L^1})}\big|f_{0,N}^{\otimes s}\big|_{\varepsilon,\beta_0}=\Big(e^{\mu_0+2\varepsilon\kappa_d|f_0|_{L^\infty L^1}}|f_0|_{0,\beta_0}\Big)^s\,,$$

the equality by (7.1.8). Given μ_0 such that $e^{\mu_0}|f_0|_{0,\beta_0}<1$, for N larger than some N_1, which we may assume to be larger than N_0, there holds $e^{\mu_0+2\varepsilon\kappa_d|f_0|_{L^\infty L^1}}|f_0|_{0,\beta_0}<1$. The above then implies

$$\sup_{N\ge N_1}\|F_{0,N}\|_{\varepsilon,\beta_0,\mu_0}\le \sup_{N\ge N_1}\sup_{1\le s\le N}\Big(e^{\mu_0+2\varepsilon\kappa_d|f_0|_{L^\infty L^1}}|f_0|_{0,\beta_0}\Big)^s<\infty\,,$$

which of course implies the uniform bound $\sup_{N\ge1}\|F_{0,N}\|_{\varepsilon,\beta_0,\mu_0}<\infty$.

Second step. We compute for $s\le N$:

$$\begin{aligned} f_{0,N}^{(s)}=\mathcal{Z}_N^{-1}f_{0,N}^{\otimes s}\int_{\mathbf{R}^{2d(N-s)}}\exp\Big(-\sum_{s+1\le i<j\le N}\beta_0\Phi_\varepsilon(x_i-x_j)-\sum_{i\le s\le s+1\le j}\beta_0\Phi_\varepsilon(x_i-x_j)\Big)\\ \times\prod_{s+1\le i\le N} f_0(z_i)\,dZ_{(s+1,N)}\,,\end{aligned}$$

and deduce, by the symmetry property (7.1.11),

$$f_{0,N}^{(s)}=\mathcal{Z}_N^{-1}f_{0,N}^{\otimes s}\Big(\mathcal{Z}_{N-s}-\mathcal{Z}^\flat_{(s+1,N)}\Big) \tag{7.1.13}$$

with the notation

$$\begin{aligned}\mathcal{Z}^\flat_{(s+1,N)}=\int_{\mathbf{R}^{2d(N-s)}}\Big(1-\exp\Big(-\sum_{i\le s\le s+1\le j}\beta_0\Phi_\varepsilon(x_i-x_j)\Big)\Big)\\ \times\exp\Big(-\sum_{s+1\le i<j\le N}\beta_0\Phi_\varepsilon(x_i-x_j)\Big)\prod_{s+1\le i\le N}f_0(z_i)\,dZ_{(s+1,N)}\,,\end{aligned}$$

so that $\mathcal{Z}^\flat_{(s+1,N)}$ is a function of X_s.

From there, the difference $f_{0,N}^{\otimes s} - f_{0,N}^{(s)}$ decomposes as a sum:

$$f_{0,N}^{\otimes s} - f_{0,N}^{(s)} = \Big(1 - \mathcal{Z}_N^{-1}\mathcal{Z}_{N-s}\Big) f_{0,N}^{\otimes s} + \mathcal{Z}_N^{-1}\mathcal{Z}^{\flat}_{(s+1,N)} f_{0,N}^{\otimes s} \,. \tag{7.1.14}$$

By Lemma 7.1.2, there holds $1 - \mathcal{Z}_N^{-1}\mathcal{Z}_{N-s} \to 0$ as $N \to \infty$, for fixed s. Since $f_\varepsilon^{\otimes s}$ is uniformly bounded in Ω_s, this implies that the first term in the right-hand side of (7.1.14) tends to 0 as $N \to \infty$, uniformly in Ω_s. Besides, by $0 \leq 1 - \exp\Big(- \beta_0 \sum_{i \leq s, s+1 \leq j} \Phi_\varepsilon(x_i - x_j)\Big) \leq \sum_{\substack{1 \leq i \leq s \\ s+1 \leq j \leq N}} 1\!\!1_{|x_i - x_j| < \varepsilon}$, we bound

$$\begin{aligned}
\mathcal{Z}^{\flat}_{(s+1,N)} \leq \sum_{1 \leq i \leq s} \int_{\mathbf{R}^{2d(N-s)}} & \Big(\sum_{s+1 \leq j \leq N} 1\!\!1_{|x_i - x_j| < \varepsilon} \Big) \\
& \times \exp\Big(- \sum_{s+1 \leq i < j \leq N} \beta_0 \Phi_\varepsilon(x_i - x_j)\Big) \prod_{s+1 \leq i \leq N} f_0(z_i)\, dZ_{(s+1,N)} \,.
\end{aligned}$$

Given $1 \leq i \leq s$, there holds by symmetry and Fubini,

$$\begin{aligned}
\int_{\mathbf{R}^{2d(N-s)}} & \Big(\sum_{s+1 \leq j \leq N} 1\!\!1_{|x_i - x_j| < \varepsilon} \Big) \exp\Big(- \sum_{s+1 \leq i < j \leq N} \beta_0 \Phi_\varepsilon(x_i - x_j)\Big) \prod_{s+1 \leq i \leq N} f_0(z_i)\, dZ_{(s+1,N)} \\
& \leq (N-s) \int_{\mathbf{R}^{2d}} 1\!\!1_{|x_i - x_{s+1}| < \varepsilon} f_0(z_{s+1}) dz_{s+1} \\
& \quad \times \int_{\mathbf{R}^{2d(N-s-1)}} \exp\Big(- \sum_{s+2 \leq i < j \leq N} \beta_0 \Phi_\varepsilon(x_i - x_j)\Big) \prod_{s+2 \leq i \leq N} f_0(z_i)\, dZ_{(s+2,N)} \\
& = (N-s) \int_{\mathbf{R}^{2d}} 1\!\!1_{|x_i - x_{s+1}| < \varepsilon} f_0(z_{s+1}) dz_{s+1} \, \times \, \mathcal{Z}_{N-s-1} \,,
\end{aligned}$$

so that

$$\mathcal{Z}^{\flat}_{(s+1,N)} \leq s(N-s)\varepsilon^d \kappa_d |f_0|_{L^\infty L^1} \mathcal{Z}_{N-s-1} \,, \tag{7.1.15}$$

where $|f_0|_{L^\infty L^1}$ denotes the $L^\infty(\mathbf{R}^d_x, L^1(\mathbf{R}^d_v))$ norm of f_0. By Lemma 7.1.2, we obtain

$$\mathcal{Z}_N^{-1}\mathcal{Z}^{\flat}_{(s+1,N)} \leq \varepsilon s \kappa_d |f_0|_{L^\infty L^1} \big(1 - \varepsilon \kappa_d |f_0|_{L^\infty L^1}\big)^{-(s+1)} \,,$$

and the upper bound tends to 0 as $N \to \infty$, for fixed s. This implies convergence to 0, uniformly in Ω_s, of the second term in the right-hand side of (7.1.14).

We thus proved the uniform convergence $f_{0,N}^{(s)} - f_{0,N}^{\otimes s} \to 0$ in Ω_s. Since $\exp(\beta E_\varepsilon(Z_s)) \to 1$ locally uniformly in Ω_s (*not* uniformly in Ω_s), the convergence $f_{0,N}^{\otimes s} \to f_0^{\otimes s}$ holds locally uniformly in Ω_s. We conclude that $f_{0,N}^{(s)}$ converges locally uniformly to tensor products $f_0^{\otimes s}$ in Ω_s.

Third step. The bound (7.1.2) is a direct consequence of the corresponding bound for $F_{0,N}$, proved in the first step, since $0 \leq \widetilde{f}_{0,N}^{(s)} \leq f_{0,N}^{(s)}$. The fact that the truncated marginals converge is due to the Lemma 7.1.3, stated and proved below.

By the normalization condition in (7.1.6), the tensor products are marginals:

$$\int_{\mathbf{R}^{2d}} f_0^{\otimes(s+1)}(Z_s, z_{s+1}) dz_{s+1} = f_0^{\otimes s}(Z_s) \int_{\mathbf{R}^{2d}} f_0(z_{s+1}) dz_{s+1} = f_0^{\otimes s}(Z_s) \,.$$

This verifies (7.1.1), and concludes the proof. $\square$

Lemma 7.1.3. — *Given $\widetilde F_{0,N} = (\widetilde f^{(s)}_{0,N})_{1\leq s\leq N}$ satisfying* (7.1.2) *and* (7.1.3) *from Definition* 7.1.1, *with associated family $F_{0,N} = (f^{(s)}_{0,N})_{1\leq s\leq N}$ of untruncated marginals:*

$$f^{(s)}_{0,N}(Z_s) = \int_{\mathbf{R}^{2d(N-s)}} f^{(N)}_{0,N}(Z_N)dZ_{(s+1,N)}\,, \quad 1\leq s<N\,, \quad Z_s\in\Omega_s\,, \qquad \widetilde f^{(N)}_{0,N} = f^{(N)}_{0,N}\,, \tag{7.1.16}$$

there holds the convergence

$$f^{(s)}_{0,N} - \widetilde f^{(s)}_{0,N} \longrightarrow 0\,, \qquad \textit{for fixed } s\geq 1\,,\ \textit{as } N\to\infty \textit{ with } N\varepsilon^{d-1}\equiv 1\,,\ \textit{uniformly in } \Omega_s\,.$$

Proof. — We apply identity (5.1.1) from Lemma 5.1.1 to $f^{(N)}_{0,N}$, and obtain after integration in the velocity variables

$$f^{(s)}_{0,N}(Z_s) - \widetilde f^{(s)}_{0,N}(Z_s) = \sum_{m=1}^{N-s} C^m_{N-s}\int_{\Delta_m(X_s)\times\mathbf{R}^{dm}} \widetilde f^{(s+m)}_{0,N}(Z_{s+m})dZ_{(s+1,s+m)}\,. \tag{7.1.17}$$

Then, denoting $C_0 = \sup_{M\geq 1}\|F_{0,M}\|_{\varepsilon,\beta_0,\mu_0}$, a finite number by assumption, from

$$\begin{aligned} f^{(s+m)}_{0,N}(Z_{s+m}) &\leq \exp\big(-\mu_0(s+m) - \beta_0 E_\varepsilon(Z_{s+m})\big)C_0 \\ &\leq \exp\Big(-\mu_0(s+m) - (\beta_0/2)\sum_{1\leq i\leq s}|v_i|^2\Big)C_0\,, \end{aligned}$$

we deduce, first by integrating the velocity gaussians and then by using the cluster bound (5.1.2) in Lemma 5.1.1 with $\zeta=\varepsilon^{-d}$, the bound

$$\begin{aligned} \int_{\Delta_m(X_s)\times\mathbf{R}^{dm}} f^{(s+m)}_{0,N}(Z_{s+m})dZ_{(s+1,s+m)} &\leq (2\pi/\beta_0)^{md/2}e^{-\mu_0(s+m)}C_0\int_{\Delta_m(X_s)} dX_{(s+1,s+m)} \\ &\leq m!(2\pi/\beta_0)^{md/2}\varepsilon^{md}e^{(\kappa_d-\mu_0)(s+m)}C_0\,. \end{aligned}$$

If $2\varepsilon e^{\kappa_d-\mu_0}(2\pi/\beta_0)^{d/2}<1$, then

$$\sum_{m=1}^{N-s} C^m_{N-s} m!(2\pi/\beta_0)^{md/2}\varepsilon^{md}e^{(\kappa_d-\mu_0)(s+m)} \leq \sum_{m\geq 1}\big(2\varepsilon e^{\kappa_d-\mu_0}(2\pi/\beta_0)^{d/2}\big)^m \longrightarrow 0$$

as $\varepsilon\to 0$, implying $f^{(s)}_{0,N} - \widetilde f^{(s)}_{0,N}\longrightarrow 0$ for fixed s, uniformly in Ω_s. □

Remark 7.1.4. — *We can reproduce the above proof in the case of a time-dependent family of bounded marginals, i.e., $F_N\in\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$, with $\sup_{N\geq 1}|||F_N|||_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}<\infty$, with the notation of Definition 6.1.1. This gives uniform convergence to zero, in time $t\in[0,T]$ and in space $X_s\in\Omega_s$, of the difference between truncated and untruncated marginals: $\widetilde f^{(s)}_N - f^{(s)}_N\to 0$.*

We now give the generalization of Proposition 7.1.2 that will be useful in the proof of Proposition 7.1.1. Let $\mathcal{P}=\mathcal{P}(\Omega_1)$ be the set of continuous densities of probability in Ω_1 :

$$\mathcal{P} = \Big\{h\in C^0(\Omega_1;\mathbf{R})\,, \quad h\geq 0\,, \quad \int_{\mathbf{R}^{2d}} h(z)dz = 1\Big\}\,. \tag{7.1.18}$$

Let π be a probability measure on $\mathcal{P}$, such that, for some $\beta_0>0$ and some $\mu_0\in\mathbf{R}$,

$$\operatorname{supp}\pi\subset\Big\{h\in\mathcal{P}\,, \quad |h|_{0,\beta_0}\leq e^{-\mu_0}\Big\}\,. \tag{7.1.19}$$

Next we define

$$\pi^{(s)} := \int_{\mathcal{P}} h^{\otimes s}d\pi(h)\,. \tag{7.1.20}$$

In the case that $\pi = \delta_{f_0}$, then (7.1.20) reduces to the tensor product (7.1.5)-(7.1.6). We let

$$h_N^{\otimes s} := \exp\Big(- \beta_0 \sum_{1 \le i < j \le s} \Phi_\varepsilon(x_i - x_j)\Big) h^{\otimes s}\,, \quad 2 \le s \le N\,, \quad h_N^{\otimes 1} = h \in \mathcal{P}\,,$$

generalizing (7.1.7), and

$$\mathcal{Z}_N := \int_{\mathbf{R}^{2dN}} h_N^{\otimes N}(Z_N)\, dZ_N\,, \quad h \in \mathcal{P}\,, \tag{7.1.21}$$

generalizing (7.1.9).

The following result is an obvious generalization of Lemma 7.1.2.

Lemma 7.1.5. — *Given π satisfying* (7.1.19) *and $h \in supp\,\pi$, the family of partition functions $\mathcal{Z}_s$ defined in* (7.1.21) *satisfies for $1 \le s \le N$ the bound*

$$1 \le \mathcal{Z}_N^{-1} \mathcal{Z}_{N-s} \le \big(1 - \varepsilon C_d e^{-\mu_0} \beta_0^{-1/2}\big)^{-s}\,,$$

where C_d depends only on d.

The next statement generalizes Proposition 7.1.2. Its proof is an immediate extension of the proof of Proposition 7.1.2 thanks to the dominated convergence theorem, using the obvious bound $h_N^{\otimes s} \le e^{-s\mu_0}$.

Proposition 7.1.3. — *Given π satisfying* (7.1.19)*, let*

$$\pi_N^{(N)} := \int_{\mathcal{P}} \mathcal{Z}_N^{-1} h_N^{\otimes N} d\pi(h)\,. \tag{7.1.22}$$

Then, families $(\pi_N^{(s)})_{1 \le s \le N}$ of truncated marginals of $\pi_N^{(N)}$, as defined in (7.1.3)*, satisfy*

$$\sup_{N \ge 1} \|(\pi_N^{(s)})_{1 \le s \le N}\|_{\varepsilon, \beta_0, \mu_0'} \le 1 \tag{7.1.23}$$

for any $\mu_0' < \mu_0$, with β_0 and μ_0 from (7.1.19)*. The data $(\pi^{(s)})_{s \ge 1}$, with $\pi^{(s)}$ defined in* (7.1.20)*, is admissible Boltzmann initial data associated with that family.*

By Proposition 7.1.2, tensor products $(f_0^{\otimes s})_{s \ge 1}$, with f_0 satisfying (7.1.6), are admissible Boltzmann data. More generally, by Proposition 7.1.3, the convex hull of the set of tensor products, in the sense of (7.1.19)-(7.1.20), is included in the set of admissible Boltzmann data. We finally show the converse: all admissible Boltzmann data belong to the convex hull of tensor products.

We first remark that given a Boltzmann datum F_0, and an associated BBGKY datum $F_{0,N}$, there holds

$$\|F_0\|_{0, \beta_0, \mu_0} < \infty\,, \tag{7.1.24}$$

with β_0 and μ_0 as in (7.1.2).

Indeed, let $C_0 = \sup_{N \ge 1} \|F_{0,N}\|_{\varepsilon, \beta_0, \mu_0} < \infty$. Given s and $Z_s \in \Omega_s$, for ε small enough, $\Phi_\varepsilon(x_i - x_j) = 0$. Besides, by (7.1.4) there holds the pointwise convergence $f_{0,N}^{(s)}(Z_s) \to f_0^{(s)}(Z_s)$. Hence taking the limit $\varepsilon \to 0$ in the left-hand side of the inequality $e^{s\mu_0 + \beta_0 E_\varepsilon(Z_s)} |f_{0,N}^{(s)}(Z_s)| \le C_0$, we find (7.1.24).

The Hewitt-Savage theorem reveals the specific role played by tensor products: the set of families $F_0 = (f_0^{(s)})_{s \ge 1}$ of marginals (7.1.1) satisfying the uniform bound (7.1.24) is the convex hull of tensorized initial data, as described in the following statement.

Proposition 7.1.4. — *Given $F_0 = (f_0^{(s)})_{s\geq 1}$ a family of marginals* (7.1.1) *satisfying the uniform bound* (7.1.24) *with constants $\beta_0 > 0$ and $\mu_0 \in \mathbf{R}$, there exists a probability measure π over the set $\mathcal{P}$ of continuous densities of probability over Ω_1, defined in* (7.1.18)*, with*

$$\operatorname{supp} \pi \subset \big\{ g \in \mathcal{P},\ |g|_{0,\beta_0} \leq e^{-\mu_0} \big\}\,, \tag{7.1.25}$$

such that the following representation holds:

$$f_0^{(s)} = \int_{\mathcal{P}} g^{\otimes s} d\pi(g)\,, \qquad s \geq 1\,. \tag{7.1.26}$$

Proof. — Given a family F_0 satisfying (7.1.1)-(7.1.24), the existence of π satisfying (7.1.26) is granted by the Hewitt-Savage theorem [**23**]. The goal is then to prove the inclusion (7.1.25). Assume by contradiction that, for some $\alpha > 0$,

$$\pi(A_\alpha) = \kappa_\alpha > 0\,, \quad \text{where } A_\alpha := \big\{ g \in \mathcal{P}(\mathbf{R}^{2d}), \quad |g|_{0,1,\beta_0} \geq e^{-\mu_0} + \alpha \big\}\,. \tag{7.1.27}$$

We then have by (7.1.26)

$$f_0^{(s)} \geq \int_{A_\alpha} g^{\otimes s} d\pi(g),$$

hence by $f_0^{(s)} \leq e^{-s\mu_0} \|F_0\|_{0,\beta_0,\mu_0}$, we infer that $\|F_0\|_{0,\beta_0,\mu_0} \geq \kappa_\alpha (1 + \alpha e^{\mu_0})^s$, which cannot hold for some $\alpha > 0$ and all s, since $1 + \alpha e^{\mu_0} > 1$. Hence (7.1.27) does not hold, which proves the result. □

Proof of Proposition 7.1.1. — We already observed in (7.1.24) that admissible Boltzmann data are bounded families of marginals. Conversely, given a bounded family of marginals F_0, by Proposition 7.1.4 representation (7.1.26) holds. Then, by Proposition 7.1.3, F_0 is an admissible Boltzmann datum. This proves Proposition 7.1.1. □

Combining Propositions 7.1.1 and 7.1.4, we see that all admissible Boltzmann data are built on tensor products, in the sense that given an admissible Boltzmann datum, representation (7.1.26) holds for some π satisfying (7.1.25).

7.2. Main result: Convergence of the BBGKY hierarchy to the Boltzmann hierarchy

7.2.1. Statement of the result. —

Our main result is a *weak convergence* result, in the sense of convergence of observables, or averages with respect to the momentum variables. Moreover, since the marginals are defined in Ω_s, we must also eliminate, in the convergence, the diagonals in physical space. Let us give a precise definition of the convergence we shall be considering.

***Definition 7.2.1* (Convergence).** — *Given a sequence $(h_N^s)_{1\leq s\leq N}$ of functions $h_N^s \in C^0(\Omega_s;\mathbf{R})$, a sequence $(h^s)_{s\geq 1}$ of functions $h^s \in C^0(\Omega_s;\mathbf{R})$, we say that (h_N^s) converges on average and locally uniformly outside the diagonals to (h^s), and we denote*

$$(h_N^s)_{1\leq s\leq N} \stackrel{\sim}{\longrightarrow} (h^s)_{1\leq s}\,,$$

when for any fixed s, any test function $\varphi_s \in \mathcal{C}_c^\infty(\mathbf{R}^{ds};\mathbf{R})$, there holds

$$I_{\varphi_s}\big(h_N^s - h^s\big)(X_s) := \int_{\mathbf{R}^{ds}} \varphi_s(V_s)\big(h_N^s - h^s\big)(Z_s) dV_s \longrightarrow 0\,, \quad \text{as } N \to \infty\,,$$

locally uniformly in $\big\{ X_s \in \mathbf{R}^{ds},\ x_i \neq x_j \text{ for } i \neq j \big\}$.

With regard to spatial variables, this notion of convergence is similar to the convergence in the sense of Chacon.

We remark that local uniform convergence in Ω_s implies convergence in the sense of Definition 7.2.1:

Lemma 7.2.2. — *Given* $(f_N^{(s)})_{1\leq s\leq N}$ *a bounded sequence in* $\mathbf{X}_{\varepsilon,\boldsymbol{\beta},\boldsymbol{\mu}}$ *with the notation of Definition* 6.1.1*, if* $f_N^{(s)} \to f^{(s)}$ *for fixed* s*, uniformly in* $t \in [0,T]$ *and locally uniformly in* Ω_s*, then* $f_N^{(s)} \overset{\sim}{\longrightarrow} f^{(s)}$*, uniformly in* $t \in [0,T]$.

Proof. — Let K_s be compact in $\big\{X_s \in \mathbf{R}^{ds},\ x_i \neq x_j \text{ for } i \neq j\big\}$. There holds

$$\big|I_{\varphi_s}\big(f_N^{(s)} - f^{(s)}\big)(X_s)\big| \leq \|\varphi_s\|_{L^1(\mathbf{R}^d)} \sup_{V_s \in \operatorname{supp}\varphi_s} \big|\big(f_N^{(s)} - f^{(s)}\big)(X_s,V_s)\big|\,.$$

The set $K_s \times \operatorname{supp}\varphi_s$ is compact in Ω_s. Hence the above upper bound converges to 0 as $N \to \infty$, uniformly in $[0,T] \times K_s$. $\square$

We can now state our main result:

***Theorem 4* (Convergence)**. — *Given a potential that satisfies Assumption* 1.2.1 *stated page* 3*, given* F_0 *admissible Boltzmann datum associated with a family* $(\widetilde{F}_{0,N})_{N\geq 1}$ *of BBGKY data, denoting* $\widetilde{F}_N$ *the unique mild solution to the BBGKY hierarchy* (4.5.2) *with initial datum* $\widetilde{F}_{0,N}$*, given by Theorem* 2*, and* F *the unique mild solution to the Boltzmann hierarchy* (6.0.2) *with initial datum* F_0*, given by Theorem* 3*, there holds the convergence*

$$\widetilde{F}_N \overset{\sim}{\longrightarrow} F\,,$$

uniformly on $[0,T]$*, for any common existence time* $T > 0$.

In particular, if the initial data $\widetilde{F}_{0,N}$ *is asymptotically tensorized, meaning that* $F_0 = (f_0^{(s)})_{s\geq 1}$ *with* $f_0^{(s)}(t,Z_s) = \prod_{i=1}^{s} f_0(t,z_i)$ *then the first marginal* $f_N^{(1)}$ *converges to the solution* f *of the Boltzmann equation* (1.3.1) *with initial data* f_0.

Solutions to the Boltzmann hierarchy issued from tensorized initial data are themselves tensorized. For such data, the Boltzmann hierarchy then reduces to the nonlinear Boltzmann equation (1.3.1), and Theorem 4 describes an asymptotic form of propagation of chaos, in the sense that an initial property of independence is propagated in time, in the thermodynamical limit. This corresponds to Theorem 1 stated in the Introduction.

The results in this chapter imply the following Corollary to Theorem 4.

Corollary 7.2.3. — *Let* F_0 *be a family of marginals* (7.1.1) *satisfying the uniform bound* (7.1.24)*, and* F *be the solution to the Boltzmann hiearchy issued from* F_0*, as given in Theorem* 3*. There exists a family of solutions* $\widetilde{F}_N$ *to the BBGKY hierarchy and* F_N *an associated family of untruncated marginals, such that*

$$\widetilde{F}_N \overset{\sim}{\longrightarrow} F \quad \textit{and} \quad F_N \overset{\sim}{\longrightarrow} F\,,$$

uniformly on $[0,T]$*, for any common existence time* T.

Proof. — By Proposition 7.1.1, the family F_0 is an admissible Boltzmann datum. Denoting $\widetilde{F}_{0,N}$ an associated BBGKY datum, let $T > 0$ be an existence time for the BBGKY hierarchy $\widetilde{F}_N$ with datum $\widetilde{F}_{0,N}$, given by Theorem 2. By Theorem 4 the convergence $I_{\varphi_s}\big(\widetilde{f}_N^{(s)} - f^{(s)}\big) \to 0$ holds uniformly in $[0,T]$ and locally uniformly in Ω_s.

Then, by Lemma 7.1.3 and Remark 7.1.4, there holds $f_N^{(s)} - \widetilde{f}_N^{(s)} \to 0$, for fixed s, uniformly in $[0,T]\times\Omega_s$. By Lemma 7.2.2, this implies $I_{\varphi_s}\big(f_N^{(s)} - \widetilde{f}_N^{(s)}\big) \to 0$, uniformly in $[0,T]$ and locally uniformly in Ω_s.

We conclude that $f_N^{(s)} \stackrel{\sim}{\longrightarrow} f^{(s)}$, uniformly in $[0,T]$. □

7.2.2. About the proof of Theorem 4: outline of chapters 8, 9 and 10. —

The formal derivation presented in Chapter 2 (in the case of hard spheres, but which could easily be adapted to our case) fails because of a number of incorrect arguments:
- Since mild solutions to the BBGKY hierarchy are defined by the Duhamel formula (4.5.2) where the solution itself occurs in the source term, we need some precise information on the convergence to take limits directly in (4.5.2).
- The irreversibility inherent to the Boltzmann hierarchy appears in the limiting process as an arbitrary choice of the time direction (encoded in the distinction between pre-collisional and post-collisional particles), and more precisely as an arbitrary choice of the initial time, which is the only time for which one has a complete information on the family of marginals $F_{0,N}$. This specificity of the initial time does not appear clearly in (4.5.2).
- The heuristic argument which allows to neglect pathological trajectories requires to be quantified. These are
  - either trajectories for which the reduced dynamics with s-particles does not coincide with the free transport ($\mathbf{H}_s \neq \mathbf{S}_s$),
  - or trajectories for which some of the (localized) interactions involve at least three particles (so that the scattering described in Chapter 3 does not apply).

  Indeed we have more or less to repeat the operation infinitely many times, since mild solutions are defined by a loop process.
- Because of the conditioning by the energy $E_\varepsilon(Z_N)$, the initial data are not so smooth. The operations such as infinitesimal translations on the arguments require therefore a careful treatment.

To overcome the two first difficulties, the idea is to start from the iterated Duhamel formula, which allows to express any marginal $\widetilde{f}_N^{(s)}(t, Z_s)$ in terms of the initial data $\widetilde{F}_{0,N}$. By successive integrations in time, we have indeed the following representation of $\widetilde{f}_N^{(s)}$:

$$\widetilde{f}_N^{(s)}(t) = \sum_{n=0}^{\infty} \sum_{M_n} \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \mathbf{H}_s(t-t_1)\mathcal{C}_{s,s+m_1}\mathbf{H}_{s+m_1}(t_1-t_2)\mathcal{C}_{s+m_1,s+m_2}\dots \dots \mathbf{H}_{s+m_n}(t_n)\widetilde{f}_N^{(s+m_n)}(0)\,dt_n\dots dt_1 \tag{7.2.1}$$

where by convention $\widetilde{f}_N^{(j)}(0) \equiv 0$ for $j > N$, and the sum over $M_n := (m_1, \dots, m_n)$ is restricted to the range $1 \leq m_{i+1} \leq N - s - m_i$ with $m_0 := 0$.

Using a **dominated convergence argument**, we shall first reduce (in Chapter 8) to the study of a functional
- which involves only the superdiagonal part of the collision operator (i.e. terms of the type $\mathcal{C}_{j,j+1}$),
- defined as a finite sum of terms (independent of N),
- where the energies of the particles are assumed to be bounded,

– and where the collision times are supposed to be well separated (namely $|t_j - t_{j+1}| \geq \delta$).
The reason for the two last assumptions is essentially technical, and will appear more clearly in the next step.

The heart of the proof, in Chapter 9, is then to prove the term by term convergence, dealing with pathological trajectories. Let us recall that each collision term is defined as an integral with respect to positions and velocities. The main idea consists then in proving that we cannot build pathological trajectories if we exclude at each step a small domain of integration. The explicit construction of this "bad set" lies on

– a very simple geometrical lemma which ensures that two particles of size ε will not collide in the future provided that their relative velocity does not belong to a small subset of $\mathbf{R}^d$ (see Lemma 9.1.3),
– scattering estimates which tell us how these properties of the transport are modified when a particle is deviated by a collision (see Lemma 9.1.4).

This construction, which is the technical part of the proof, will be detailed in Chapter 9. The conclusion of the convergence proof is presented in Chapter 10.

CHAPTER 8

REDUCTIONS VIA DOMINATED CONVERGENCE

The goal of this chapter is to use dominated convergence arguments to reduce the proof of Theorem 4 to the term-by-term convergence of some functionals involving a finite (uniformly bounded) number of marginals (Sections 8.1 and 8.2). In order to further simplify the convergence analysis, we shall modify these functionals by eliminating some small domains of integration in phase space corresponding to pathological dynamics, namely large energies in Section 8.3 and clusters of collision times in Section 8.4.

We consider therefore families of initial data: Boltzmann initial data $F_0 = (f_0^{(s)})_{s\in\mathbf{N}}$ such that

$$|\!|\!| F_0 |\!|\!|_{0,\beta_0,\mu_0} = \sup_{s\in\mathbf{N}} \sup_{Z_s} \big(\exp(\beta_0 E(Z_s) + \mu_0 s) f_0^{(s)}(Z_s) \big) < +\infty$$

and for each N, BBGKY initial data $\widetilde F_{N,0} = (\widetilde f_{N,0}^{(s)})_{1\le s\le N}$ such that

$$\sup_N |\!|\!| \widetilde F_{N,0} |\!|\!|_{\varepsilon,\beta_0,\mu_0} = \sup_N \sup_{s\le N} \sup_{Z_s} \big(\exp(\beta_0 E_\varepsilon(Z_s) + \mu_0 s) \widetilde f_{N,0}^{(s)}(Z_s) \big) < +\infty\,.$$

We then associate the respective unique mild solutions (constructed in Theorems 2 and 3 in Chapter 6) of the hierarchies

$$f^{(s)}(t) = \mathbf{S}_s(t) f_0^{(s)} + \int_0^t \mathbf{S}_s(t-\tau)\mathcal{C}^0_{s,s+1} f^{(s+m)}(\tau)\, d\tau$$

and

$$\widetilde f_N^{(s)}(t) = \mathbf{H}_s(t)\widetilde f_{N,0}^{(s)} + \sum_{m=1}^{N-s}\int_0^t \mathbf{H}_s(t-\tau)\mathcal{C}_{s,s+m}\widetilde f_N^{(s+m)}(\tau)\, d\tau\,. \tag{8.0.1}$$

In terms of the initial datum, they can be rewritten

$$f^{(s)}(t) = \sum_{n=0}^\infty \int_0^t\int_0^{t_1}\dots\int_0^{t_{n-1}} \mathbf{S}_s(t-t_1)\mathcal{C}^0_{s,s+1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}^0_{s+1,s+2}\dots$$
$$\dots \mathbf{S}_{s+n}(t_n) f^{(s+m_n)}(0)\, dt_n\dots dt_1$$

and

$$\widetilde f_N^{(s)}(t) = \sum_{n=0}^\infty \sum_{m_1,\dots,m_n} \int_0^t\int_0^{t_1}\dots\int_0^{t_{n-1}} \mathbf{H}_s(t-t_1)\mathcal{C}_{s,s+m_1}\mathbf{H}_{s+m_1}(t_1-t_2)\mathcal{C}_{s+m_1,s+m_2}\dots$$
$$\dots \mathbf{H}_{s+m_n}(t_n)\widetilde f_N^{(s+m_n)}(0)\, dt_n\dots dt_1\,.$$

The observables we are interested in therefore involve infinite sums

– because of multiple collisions in the BBGKY hierarchy, i.e. of collision terms $\mathcal{C}_{s,s+m}$ where m is not uniformly bounded,

– and because there may be infinitely many collision times (n is unbounded).

8.1. Reduction to first-order collision terms for the BBGKY hierarchy

We will first prove that the estimates obtained in Chapters 5 and 6 enable us to reduce the study of the BBGKY hierarchy to the equation

$$\widetilde{g}_N^{(s)}(t,Z_s) = \mathbf{H}_s(t)\widetilde{f}_N^{(s)}(0,Z_s) + \int_0^t \mathbf{H}_s(t-\tau)\mathcal{C}_{s,s+1}\widetilde{g}_N^{(s+1)}(\tau,Z_s)\,d\tau\,, \qquad 1\leq s\leq N-1\,. \tag{8.1.2}$$

Estimate (5.3.1) in Proposition 5.3.1 shows indeed that higher-order collisions are negligible in the Boltzmann-Grad limit. For the solution to the BBGKY hierarchy, this translates as follows.

Proposition 8.1.1. — *Let $\beta_0>0$ and μ_0 be given. Then with the same notation as Theorem* 2, *in the Boltzmann-Grad scaling $N\varepsilon^{d-1}\equiv 1$, any family of initial marginals $\widetilde{F}_N(0) = \big(\widetilde{f}_N^{(s)}(0)\big)_{1\leq s\leq N}\in \mathbf{X}_{\varepsilon,\beta_0,\mu_0}$ gives rise to a unique solution $\widetilde{G}_N\in \mathbf{X}^{\lambda,T}_{\varepsilon,\beta_0,\mu_0}$ of* (8.1.2) *and there holds the bound*

$$\|\widetilde{G}_N\|_{\mathbf{X}^{\lambda,T}_{\varepsilon,\beta_0,\mu_0}} \leq C\|\widetilde{F}_N(0)\|_{\varepsilon,\beta_0,\mu_0}\,.$$

Besides, the solution $\widetilde{G}_N$ to the modified hierarchy (8.1.2) *is asymptotically close to the solution $\widetilde{F}_N$ to the BBGKY hierarchy* (6.0.1)*:*

$$\|\widetilde{G}_N-\widetilde{F}_N\|_{\mathbf{X}^{\lambda,T}_{\varepsilon,\beta_0,\mu_0}} \leq \varepsilon\tilde{C}\|\widetilde{F}_N(0)\|_{\varepsilon,\beta_0,\mu_0} \tag{8.1.3}$$

for some $\tilde{C}>0$.

In particular, given an existence time T for the BBGKY hierarchy, in the sense of Theorem 2, then T is an existence time for the modified hierarchy (8.1.2), in the sense of Proposition 8.1.1.

Proof. — From Lemma 6.2.1, we deduce the existence and uniqueness result for (8.1.2), and the bound for $\widetilde{G}_N$, in the same way that Lemma 6.2.2 implies Theorem 2. Notice that an existence time for the BBGKY hierarchy is an existence time for the modified hierarchy, since the bound (6.2.8) is better than (6.1.6).

We turn to the proof of (8.1.3). There holds

$$\begin{aligned}\|\widetilde{G}_N-\widetilde{F}_N\|_{\mathbf{X}^{\lambda,T}_{\varepsilon,\beta_0,\mu_0}} \leq{}& \Big\|\int_0^t \Big(\mathbf{H}_s(t-t')\mathcal{C}_{s,s+1}(\widetilde{g}_N^{(s+1)}-\widetilde{f}_N^{(s+1)})(t')\Big)_{1\leq s\leq N}dt'\Big\|_{\mathbf{X}^{\lambda,T}_{\varepsilon,\beta_0,\mu_0}}\\ &+\Big\|\int_0^t\Big(\mathbf{H}_s(t-t')\sum_{2\leq m\leq N-s}\mathcal{C}_{s,s+m}f_N^{(s+m)}(t')\Big)_{1\leq s\leq N}dt'\Big\|_{\mathbf{X}^{\lambda,T}_{\varepsilon,\beta_0,\mu_0}}\,.\end{aligned}$$

With (6.2.8), this implies

$$\|\widetilde{G}_N-\widetilde{F}_N\|_{\mathbf{X}^{\lambda,T}_{\varepsilon,\beta_0,\mu_0}} \leq c_0\Big\|\int_0^t\Big(\mathbf{H}_s(t-t')\sum_{2\leq m\leq N-s}\mathcal{C}_{s,s+m}f_N^{(s+m)}(t')\Big)_{1\leq s\leq N}dt'\Big\|_{\mathbf{X}^{\lambda,T}_{\varepsilon,\beta_0,\mu_0}}\,,$$

with $c_0 := \big(1-\bar{c}(\beta_0,\mu_0,\lambda,T)\big)^{-1}$, which is indeed strictly positive by assumption. We conclude as in the proof of Lemma 6.2.2. □

From now on we therefore shall concentrate on equation (8.1.2) instead of (8.0.1).

8.2. Reduction to a finite number of collision times

By successive integrations in time of (8.1.2), we get a representation of $\widetilde{g}_N^{(s)}$ in terms of the initial datum $\widetilde{f}_{N,0}^{(s+n)}$, for all n such that $s+n \leq N$:

$$\widetilde{g}_N^{(s)}(t) = \sum_{n=0}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \mathbf{H}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots \dots \mathbf{H}_{s+n}(t_n)\widetilde{f}_N^{(s+n)}(0)\, dt_n \dots dt_1 \tag{8.2.1}$$

where by convention $\widetilde{f}_{N,0}^{(j)} \equiv 0$ for $j > N$.

Similarly, for mild solutions to the Boltzmann hierarchy, we have

$$f^{(s)}(t) = \sum_{n=0}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \mathbf{S}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots \dots \mathbf{S}_{s+n}(t_n)f_0^{(s+n)}(0)\, dt_n \dots dt_1 \,. \tag{8.2.2}$$

Due to the uniform bounds derived in Chapter 6, the dominated convergence theorem implies that it is enough to consider finite sums

$$\begin{aligned} \widetilde{g}_N^{(s,n)}(t) &= \sum_{k=0}^{n} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{H}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots \\ &\qquad \dots \mathbf{H}_{s+k}(t_k)\widetilde{f}_N^{(s+k)}(0)\, dt_k \dots dt_1 \\ f^{(s,n)}(t) &= \sum_{k=0}^{n} \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} \mathbf{S}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots \\ &\qquad \dots \mathbf{S}_{s+k}(t_k)f_0^{(s+k)}(0)\, dt_k \dots dt_1 \end{aligned} \tag{8.2.3}$$

and therefore to study the term-by-term convergence (for any fixed k), as expressed by the following statement.

Proposition 8.2.1. — *The following estimates*

$$\left|\widetilde{g}_N^{(s)}(t) - \widetilde{g}_N^{(s,n)}(t)\right| \leq C\left(\frac{2}{3}\right)^n \|\widetilde{F}_{N,0}\|_{\varepsilon,\beta_0,\mu_0} e^{s\mu(T)}\,,$$
$$\left|f^{(s)}(t) - f^{(s,n)}(t)\right| \leq C\left(\frac{2}{3}\right)^n \|F_0\|_{0,\beta_0,\mu_0} e^{s\mu(T)}$$

hold uniformly in Z_s, N and $t \leq T$, in the Boltzmann-Grad scaling $N\varepsilon^{d-1} = 1$.

Proof. — Using the continuity estimate (6.1.6) together with the condition (6.2.19) on λ, we get

$$\Big\| \int_0^t \mathbf{H}(t-t')\mathbf{C}_N G_N(t')\, dt' \Big\|_{\varepsilon,\beta_0-\lambda t,\mu_0-\lambda t} \leq \frac{2}{3}\|G_N\|_{\mathbf{X}^{\lambda,T}_{\varepsilon,\beta_0,\mu_0}}\,. \tag{8.2.4}$$

We then deduce that

$$e^{s(\mu_0-\lambda t)}\Big\| \sum_{k=n+1}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} \mathbf{H}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots \dots \mathbf{H}_{s+k}(t_k)\widetilde{f}_N^{(s+k)}(0)\, dt_k \dots dt_1\Big\|_{L^\infty} \leq C\left(\frac{2}{3}\right)^n \|G_N\|_{\mathbf{X}^{\lambda,T}_{\varepsilon,\beta_0,\mu_0}} \tag{8.2.5}$$

Combining this estimate together with the uniform bound on $\|G_N\|_{\mathbf{X}^{\lambda,T}_{\varepsilon,\beta_0,\mu_0}}$ leads to the first statement in Proposition 8.2.1.

The second statement is established exactly in an analogous way, using estimate (5.4.2) together with the uniform bound obtained in Theorem 3. □

From now on we therefore consider the approximate observables :

$$I_{s,n}(t)(X_s) := \int \varphi_s(V_s)\widetilde{g}_N^{(s,n)}(t,Z_s)dV_s\,, \quad \text{and } I^0_{s,n}(t)(X_s) := \int \varphi_s(V_s)f^{(s,n)}(t,Z_s)dV_s\,. \tag{8.2.6}$$

8.3. Energy truncation

Recall the definitions of the Hamiltonians given in (5.2.2) and (5.2.3):

$$E_\varepsilon(Z_s) := \sum_{1\le i\le s}\frac{|v_i|^2}{2} + \sum_{1\le i<k\le s}\Phi_\varepsilon(x_i-x_k) \text{ with } \Phi_\varepsilon(x) := \Phi\left(\frac{x}{\varepsilon}\right)\,, \quad \text{and} \quad E_0(Z_s) := \sum_{1\le i\le s}\frac{|v_i|^2}{2}\,.$$

We introduce a parameter $R>0$ and define

$$\begin{aligned}
\widetilde{g}^{(s,n)}_{N,R}(t) &= \sum_{k=0}^n \int_0^t\int_0^{t_1}\dots\int_0^{t_{k-1}} \mathbf{H}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots\\
&\qquad\qquad \dots\mathbf{H}_{s+k}(t_k)\mathbb{1}_{|E_\varepsilon(Z_{s+k})|\le R^2}\widetilde{f}_N^{(s+k)}(0)\,dt_k\dots dt_1\,,\\
f_R^{(s,n)}(t) &= \sum_{k=0}^n \int_0^t\int_0^{t_1}\dots\int_0^{t_{n-1}} \mathbf{S}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\dots\\
&\qquad\qquad \dots\mathbf{S}_{s+k}(t_k)\mathbb{1}_{|E_0(Z_{s+k})|\le R^2}f_0^{(s+k)}(0)\,dt_k\dots dt_1
\end{aligned} \tag{8.3.1}$$

and the corresponding observables

$$I^R_{s,n}(t)(X_s) := \int \varphi_s(V_s)\widetilde{g}^{(s,n)}_{N,R}(t,Z_s)dV_s\,, \quad \text{and } I^0_{s,n}(t)(X_s) := \int \varphi_s(V_s)f_R^{(s,n)}dV_s\,. \tag{8.3.2}$$

Using the bounds derived in Chapters 5 and 6 we find easily that $I_{s,n}(t)-I^R_{s,n}(t)$ and $I^0_{s,n}(t)-I^{0,R}_{s,n}(t)$ can be made arbitrarily small when R is large. More precisely the following result holds.

Proposition 8.3.1. — *There are some nonnegative constants C, C' depending only on (s,n,t) such that*

$$\|I_{s,n}(t,J,M)-I^R_{s,n}(t,J,M)\|_{L^\infty(\mathbf{R}^{ds})} \le C\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}e^{-C'R^2}\|\widetilde{F}_{N,0}\|_{\varepsilon,\beta_0,\mu_0}\,,$$

and

$$\|I^0_{s,n}(t,J,M)-I^{0,R}_{s,n}(t,J,M)\|_{L^\infty(\mathbf{R}^{ds})} \le C\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}e^{-C'R^2}\|F_0\|_{0,\beta_0,\mu_0}\,.$$

Proof. — Let $\beta_0'<\beta_0$ be such that $\beta_0'-\lambda t>0$. The estimates in Chapter 6 show that

$$\begin{aligned}
|I_{s,n}(t)(X_s)-I^R_{s,n}(t)(X_s)| &\le C_n\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}\sup_{\tau\le t}\sup_{k\le n}\left\|\mathbb{1}_{|E_\varepsilon(Z_{s+k})|\ge R^2}\widetilde{f}^{(s+k)}_{N,0}\right\|_{\varepsilon,s+k,\beta_0'-c\tau}\\
&\le C\|\varphi_s\|_{L^\infty(\mathbf{R}^{ds})}e^{(\beta_0'-\beta_0)R^2}\|\widetilde{F}_N(0)\|_{\beta_0,\mu_0}
\end{aligned}$$

which proves the result, with $C'\sim\beta_0-ct$. The argument is identical for $I^0_{s,n}(t)-I^{0,R}_{s,n}(t)$. □

Remark 8.3.1. — *It is useful to notice that the collision operators preserve the bound on high energies, in the sense that*

$$\mathcal{C}_{s,s+1}\mathbb{1}_{E_\varepsilon(Z_{s+1})\le R^2} \equiv \mathbb{1}_{E_\varepsilon(Z_s)\le R^2}\,\mathcal{C}_{s,s+1}\mathbb{1}_{E_\varepsilon(Z_{s+1})\le R^2}$$
$$\mathcal{C}^0_{s,s+1}\mathbb{1}_{E(Z_{s+1})\le R^2} \equiv \mathbb{1}_{E(Z_s)\le R^2}\,\mathcal{C}^0_{s,s+1}\mathbb{1}_{E(Z_{s+1})\le R^2}\,.$$

8.4. Time separation

We choose another small parameter $\delta > 0$ and further restrict the study to the case when $t_i - t_{i+1} \ge \delta$. That is, we define

$$\mathcal{T}_n(t) := \Big\{T_n = (t_1,\dots,t_n)\,/\,t_i < t_{i-1} \ \text{ with }\ t_{n+1}=0 \ \text{ and }\ t_0 = t\Big\}\,,$$

$$\mathcal{T}_{n,\delta}(t) := \Big\{T_n \in \mathcal{T}\,/\,t_i - t_{i+1} \ge \delta\Big\}\,,$$

and

$$(8.4.1)\qquad \begin{aligned} I^{R,\delta}_{s,n}(t)(X_s) &:= \int \varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)} \mathbf{H}_s(t-t_1)\mathcal{C}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\mathcal{C}_{s+1,s+2}\\ &\qquad\dots\mathcal{C}_{s+n-1,s+n}\mathbf{H}_{s+n}(t_n-t_{n+1})\mathbb{1}_{|E_\varepsilon(Z_{s+n})|\le R^2}\widetilde{f}^{(s+n)}_{N,0}\,dT_n dV_s\,,\\ I^{0,R,\delta}_{s,n}(t)(X_s) &:= \int \varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)} \mathbf{S}_s(t-t_1)\mathcal{C}^0_{s,s+1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}^0_{s+1,s+2}\\ &\qquad\dots\mathcal{C}^0_{s+n-1,s+n}\mathbf{S}_{s+n}(t_n-t_{n+1})\mathbb{1}_{|E_0(Z_{s+n})|\le R^2}f^{(s+n)}_0\,dT_n dV_s\,. \end{aligned}$$

Again applying the continuity bounds for the transport and collision operators, the error on the functions $I^R_{s,n}(t) - I^{R,\delta}_{s,n}(t)$ and $I^{0,R}_{s,n}(t) - I^{0,R,\delta}_{s,n}(t)$ can be estimated as follows.

Proposition 8.4.1. — *There is a constant C depending only on (s,n,t), such that*

$$\|I^R_{s,n}(t) - I^{R,\delta}_{s,n}(t)\|_{L^\infty(\mathbf{R}^{ds})} \le C\delta\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}|\!|\!|\widetilde{F}_{N,0}|\!|\!|_{\varepsilon,\beta_0,\mu_0}$$

and

$$\|I^{0,R}_{s,n}(t) - I^{0,R,\delta}_{s,n}(t)\|_{L^\infty(\mathbf{R}^{ds})} \le C\delta\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}|\!|\!|F_0|\!|\!|_{0,\beta_0,\mu_0}\,.$$

8.5. Reformulation in terms of pseudo-trajectories

In the integrand of the collision operators $\mathcal{C}_{s,s+1}$ defined in (4.4.4), we now distinguish between pre- and post-collisional configurations, as we decompose

$$\mathcal{C}_{s,s+1} = \mathcal{C}^+_{s,s+1} - \mathcal{C}^-_{s,s+1}$$

where

$$(8.5.1)\qquad \mathcal{C}^\pm_{s,s+1}\widetilde{f}^{(s+1)} = \sum_{m=1}^{s}\mathcal{C}^{\pm,m}_{s,s+1}\widetilde{f}^{(s+1)}$$

the index m referring to the index of the interaction particle among the s "fixed" particles, with the notation

$$\big(\mathcal{C}^{\pm,m}_{s,s+1}\widetilde{f}^{(s+1)}\big)(Z_s) := (N-s)\varepsilon^2 \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} (\nu\cdot(v_{s+1}-v_m))_\pm \widetilde{f}^{(s+1)}(Z_s, x_m+\varepsilon\nu, v_{s+1}) \times \prod_{\substack{1\le j\le s\\ j\ne m}} 1\!\!1_{|x_j-x_{s+1}|\ge\varepsilon}\, d\nu dv_{s+1}\,,$$

the index $+$ corresponding to post-collisional configurations and the index $-$ to pre-collisional configurations, according to terminology set out in Chapter 3.

In the same way we have to decompose the Boltzmann collision operators (5.4.1) into

$$\mathcal{C}^0_{s,s+1} = \mathcal{C}^{0,+}_{s,s+1} - \mathcal{C}^{0,-}_{s,s+1}\,,$$

where the index $+$ corresponding to post-collisional configurations and the index $-$ to pre-collisional configurations. By definition of the collision cross-section in Chapter 3, we have

$$\begin{aligned}
\big(\mathcal{C}^{0,-,m}_{s,s+1} f^{(s+1)}\big)(Z_s) &:= \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} b(|v_{s+1}-v_m|,\omega) f^{(s+1)}(Z_s,x_m,v_{s+1})\, d\omega dv_{s+1}\\
&= \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} ((v_{s+1}-v_m)\cdot\nu)_+ f^{(s+1)}(Z_s,x_m,v_{s+1})\, d\nu dv_{s+1} \quad \text{and}\\
\big(\mathcal{C}^{0,+,m}_{s,s+1} f^{(s+1)}\big)(Z_s) &:= \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} b(|v_{s+1}-v_m|,\omega) f^{(s+1)}(z_1,\dots,x_m,v_m^*,\dots,z_s,x_m,v_{s+1}^*)\, d\omega dv_{s+1}\\
&= \int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d} ((v_{s+1}-v_m)\cdot\nu)_+ f^{(s+1)}(z_1,\dots,x_m,v_m^*,\dots,z_s,x_m,v_{s+1}^*)\, d\nu dv_{s+1}\,.
\end{aligned}$$

Performing the change of variables $\nu\mapsto-\nu$ in the integral defining $\mathcal{C}^{0,-,m}_{s,s+1}$, we get similar formulas as for the BBGKY collision operators.

The BBGKY and Boltzmann observables we are interested in (see the notation of Definition 7.2.1) can therefore be decomposed as

$$(8.5.2)\qquad \begin{aligned}
I^{R,\delta}_{s,n}(t,X_s) &= \sum_{n=0}^\infty \sum_{J,M} \Big(\prod_{i=1}^n j_i\Big) I^{R,\delta}_{s,n}(t,J,M)(X_s) \quad \text{and}\\
I^{0,R,\delta}_{s,n}(t,X_s) &= \sum_{n=0}^\infty \sum_{J,M} I^{0,R,\delta}_{s,n}(t,J,M)(X_s)
\end{aligned}$$

where the *elementary functionals* $I^{R,\delta}_{s,n}(t,J,M)$ are defined by

$$(8.5.3)\qquad \begin{aligned}
I^{R,\delta}_{s,n}(t,J,M)(X_s) &:= \int \varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)} \mathbf{H}_s(t-t_1)\mathcal{C}^{j_1,m_1}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\mathcal{C}^{j_2,m_2}_{s+1,s+2}\\
&\qquad\qquad \dots \mathbf{H}_{s+n}(t_n-t_{n+1}) 1\!\!1_{|E_\varepsilon(Z_{s+n})|\le R^2} \widetilde{f}^{(s+n)}_{N,0}\, dT_n dV_s\,,\\
I^{0,R,\delta}_{s,n}(t,J,M)(X_s) &:= \int \varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)} \mathbf{S}_s(t-t_1)\mathcal{C}^{0,j_1,m_1}_{s,s+1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}^{0,j_2,m_2}_{s+1,s+2}\\
&\qquad\qquad \dots \mathbf{S}_{s+n}(t_n-t_{n+1}) 1\!\!1_{|E_0(Z_{s+n})|\le R^2} f^{(s+n)}_0\, dT_n dV_s\,,
\end{aligned}$$

with

$$J := (j_1,\dots,j_n)\in\{+,-\}^n \quad\text{and}\quad M := (m_1,\dots,m_n) \quad\text{with}\quad m_i\in\{1,\dots,s+i-1\}\,.$$

Each one of the previous functionals $I_{s,n}^{R,\delta}(t,J,M)$ and $I_{s,n}^{0,R,\delta}(t,J,M)$ defined in (8.4.1) can be viewed as the observable associated with some dynamics, which of course is not the actual dynamics in physical space since
– the total number of particles is not conserved;
– the distribution does even not remain nonnegative because of the sign of loss collision operators.
This explains the terminology of "pseudo-trajectories" we choose to describe the process.

In this formulation, the characteristics associated with the operators $\mathbf{H}_{s+i}(t_i - t_{i+1})$ and $\mathbf{S}_{s+i}(t_i - t_{i+1})$ are followed *backwards* in time between two consecutive times t_{i+1} and t_i, and collision terms (associated with $\mathcal{C}_{s+i,s+i+1}$ and $\mathcal{C}^0_{s+i,s+i+1}$) are seen as *source terms*, in which, in the words of Lanford [**31**], "additional particles" are "adjoined" to the marginal.

The main heuristic idea is that for the BBGKY hierarchy, in the time interval $[t_{i+1}, t_i]$ between two collisions $\mathcal{C}_{s+i-1,s+i}$ and $\mathcal{C}_{s+i,s+i+1}$, the particles should not interact in general so trajectories should correspond to the free flow $\mathbf{S}_{s+i}$. On the other hand at a collision time t_i, the two particles in interaction may be subject to the scattering operator and thus their velocities are liable to change. This is depicted in Figure 5.

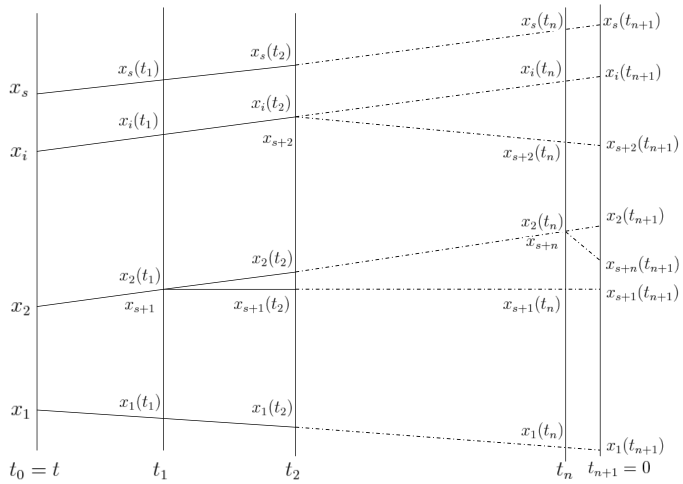

FIGURE 5. Pseudo-trajectories

At this stage however, we still cannot study directly the convergence of $I_{s,n}^{R,\delta}(t,J,M) - I_{s,n}^{0,R,\delta}(t,J,M)$ since the transport operators $\mathbf{H}_k$ do not coincide everywhere with the free transport operators $\mathbf{S}_k$, which means – in terms of pseudo-trajectories – that there are recollisions. Note that, because the interaction potential is compactly supported, recollisions happen only for characteristics such that there exist $i, j \in [1,k]$ with $i \neq j$, and $\tau > 0$ such that

$$|(x_i - \tau v_i) - (x_j - \tau v_j)| \leq \varepsilon\,.$$

We shall thus prove that these recollisions arise only for a few pathological pseudo-trajectories, which can be eliminated by additional truncations of the domains of integration. This is the goal of the next chapter.

CHAPTER 9

ELIMINATION OF RECOLLISIONS

We have seen in the previous chapter that the convergence of observables (stated in Theorem 4 in Chapter 7) reduces to the convergence to zero of the functional $I_{s,n}^{R,\delta} - I_{s,n}^{0,R,\delta}$, defined in (8.4.1), corresponding to dynamics
– involving only a finite number $s+n$ of particles,
– with bounded energies (at most R^2),
– such that the n additional particles are adjoined through binary collisions,
– at times separated at least by δ.

Let us denote, for any constant $c>0$, by $\mathcal{G}_k(c)$ the set of "good configurations" of k particles, separated by at least c through backwards transport: that is the set of $(X_k, V_k) \in \mathbf{R}^{dk} \times B_R^k$ such that the image of (X_k, V_k) by the backward free transport satisfies the separation condition

$$\forall \tau \geq 0, \quad \forall i \neq j, \quad |x_i - x_j - \tau(v_i - v_j)| \geq c\,,$$

in particular it is never collisional. We recall that $B_R^k := \Big\{ V_k \in \mathbf{R}^{dk} \,/\, |V_k| \leq R \Big\}$ and in the following we write $B_R := B_R^1$.

Our goal in the present chapter is to slightly modify (in a uniform way) the functionals $I_{s,n}^{R,\delta}$ and $I_{s,n}^{0,R,\delta}$ in order for the corresponding BBGKY pseudo-trajectories to be decomposed as a succession of free transport and binary collisions, without any recollision. Paragraph 9.1 is devoted to the statement and the proof of a geometrical proposition showing how to eliminate bad sets in phase space so that for any particle outside such bad sets, adjoined to a good configuration, the resulting configuration is again a good configuration. This is applied to the Boltzmann and BBGKY hierarchies in Paragraph 9.2.

9.1. Elimination of bad sets in phase space leading to recollisions

9.1.1. Statement of the result. — In this section we momentarily forget the BBGKY and Boltzmann hierarchies, and focus on the study of pseudo-trajectories. More precisely our aim is to show that "good configurations" are stable by adjunction of a collisional particle provided that the deflection angle and the velocity of the additional particle do not belong to a small pathological set. Furthermore the set to be excluded can be chosen in a uniform way with respect to the initial positions of the particles in a small neighborhood of any fixed "good configuration".

Notation 9.1.1. — *In all the sequel, given two parameters η_1 and η_2, we shall say that*

$$\eta_1 \ll \eta_2 \ \text{ if } \eta_1 \leq C\eta_2$$

for some large constant C which does not depend on any parameter.

Proposition 9.1.1. — *Let $a, \varepsilon_0, \eta \ll 1$ be such that*

$$a \ll \varepsilon_0 \ll \eta\delta\,. \tag{9.1.1}$$

Given $\overline{Z}_k \in \mathcal{G}_k(\varepsilon_0)$, there is a subset $\mathcal{B}_k(\overline{Z}_k)$ of $\mathbf{S}_1^{d-1} \times B_R$ of small measure

$$\big|\mathcal{B}_k(\overline{Z}_k)\big| \leq C(R)\eta^{d-1} + C(R,\Phi,\eta)\Big(\frac{a}{\varepsilon_0} + \frac{\varepsilon_0}{\delta}\Big)^{d-1}\,, \tag{9.1.2}$$

and such that good configurations close to $\overline{Z}_k$ are stable by adjunction of a collisional particle close to $\bar{x}_k$ and not belonging to $\mathcal{B}_k(\overline{Z}_k)$, in the following sense.

Consider $(\nu, v) \in (\mathbf{S}_1^{d-1} \times B_R) \setminus \mathcal{B}_k(\overline{Z}_k)$ and let Z_k be a configuration of k particles such that $V_k = \overline{V}_k$ and $|X_k - \overline{X}_k| \leq a$.

- *If $\nu \cdot (v - \bar{v}_k) < 0$ then for all $\varepsilon > 0$ sufficiently small,*

$$\forall \tau \geq 0\,, \quad \begin{cases} \forall i \neq j \in [1,k]\,, & |(x_i - \tau\bar{v}_i) - (x_j - \tau\bar{v}_j)| \geq \varepsilon\,, \\ \forall j \in [1,k]\,, & |(x_k + \varepsilon\nu - \tau v) - (x_j - \tau\bar{v}_j)| \geq \varepsilon\,. \end{cases} \tag{9.1.3}$$

Moreover after the time δ, the $k+1$ particles are in a good configuration:

$$(X_k - \delta\overline{V}_k, \overline{V}_k, x_k + \varepsilon\nu - \delta v, v) \in \mathcal{G}_{k+1}(\varepsilon_0/2)\,. \tag{9.1.4}$$

- *If $\nu \cdot (v - \bar{v}_k) > 0$ then define for $j \in [1, k-1]$*

$$(z_k^{\varepsilon *}, z^{\varepsilon *}) := \sigma_\varepsilon^{-1}\big(z_k, (x_k + \varepsilon\nu, v)\big) \quad \text{and} \quad z_j^{\varepsilon *} := (x_j - t_\varepsilon \bar{v}_j, \bar{v}_j)\,,$$

where σ_ε is the scattering operator as in Definition 3.2.1 and where t_ε denotes the scattering time between z_k and $(x_k + \varepsilon\nu, v)$. Then for all $\varepsilon > 0$ sufficiently small,

$$\forall \tau \geq 0\,, \quad \begin{cases} \forall i \neq j \in [1,k]\,, & |(x_i^{\varepsilon *} - \tau v_i^{\varepsilon *}) - (x_j^{\varepsilon *} - \tau v_j^{\varepsilon *})| \geq \varepsilon\,, \\ \forall j \in [1,k]\,, & |(x^{\varepsilon *} - \tau v^{\varepsilon *}) - (x_j^{\varepsilon *} - \tau v_j^{\varepsilon *})| \geq \varepsilon\,. \end{cases} \tag{9.1.5}$$

Moreover after the time δ, the $k+1$ particles are in a good configuration:

$$\Big(X_k^{\varepsilon *} - (\delta - t_\varepsilon)V_k^{\varepsilon *}, V_k^{\varepsilon *}, x^{\varepsilon *} - (\delta - t_\varepsilon)v^{\varepsilon *}, v^{\varepsilon *}\Big) \in \mathcal{G}_{k+1}(\varepsilon_0/2)\,. \tag{9.1.6}$$

The proof of the proposition may be found in Section 9.1.3. It relies on two elementary geometrical lemmas, stated and proved in the next section. The first one describes the bad trajectories associated to the (free) transport. The second one explains how they are modified by the scattering.

Remark 9.1.2. — *For the sake of simplicity, we have assumed that the additional particle collides with the particle k. Of course, a simple symmetry argument shows that an analogous statement holds if the particle $k+1$ is added close to any of the particles in Z_k.*

The proof of Proposition 9.1.1 shows that if $Z_k = \overline{Z}_k$ then the factor $\varepsilon_0/2$ in (9.1.4) and (9.1.6) may be replaced by ε_0. The loss comes from the fact that the set to be excluded has to be chosen in a uniform way with respect to the initial positions of the particles in a small neighborhood of $\bar{X}_k$.

9.1.2. Two geometrical lemmas. — In this section we state and prove two geometrical lemmas which though elementary, are the key to the proof of Proposition 9.1.1. Here and in the sequel, we denote by $K(y,\rho)$ the cylinder of axis $y \in \mathbf{R}^d$ and radius $\rho > 0$ and by $B_\rho(y)$ the ball centered at y of radius ρ.

Lemma 9.1.3. — *Given $\bar a > 0$ satisfying $\varepsilon \ll \bar a \ll \varepsilon_0$, consider $\bar x_1, \bar x_2$ in $\mathbf{R}^d$ such that $|\bar x_1 - \bar x_2| \geq \varepsilon_0$, and $v_1 \in B_R$. Then for any $x_1 \in B_{\bar a}(\bar x_1)$, any $x_2 \in B_{\bar a}(\bar x_2)$ and any $v_2 \in B_R$, the following results hold.*

- *If $v_1 - v_2 \notin K(\bar x_1 - \bar x_2, 6R\bar a/\varepsilon_0)$, then*
$$\forall \tau \geq 0\,, \quad |(x_1 - v_1\tau) - (x_2 - v_2\tau)| > \varepsilon\,;$$
- *If $v_1 - v_2 \notin K(\bar x_1 - \bar x_2, 6R\varepsilon_0/\delta)$*
$$\forall \tau \geq \delta\,, \quad |(x_1 - v_1\tau) - (x_2 - v_2\tau)| > \varepsilon_0\,.$$

Proof. — • Assume that there exists τ_* such that
$$|(x_1 - v_1\tau) - (x_2 - v_2\tau)| \leq \varepsilon\,.$$
Then, by the triangular inequality and provided that ε is sufficiently small,
$$|(\bar x_1 - \bar x_2) - \tau_*(v_1 - v_2)| \leq \varepsilon + 2\bar a \leq 3\bar a\,.$$
This means that $(v_1 - v_2)$ belongs to the cone of vertex 0 based on the ball centered at $\bar x_1 - \bar x_2$ and of radius $3\bar a$, which is a cone of solid angle $(3\bar a/|\bar x_1 - \bar x_2|)^{d-1}$ (since $\bar a \ll \varepsilon_0$).

The intersection of this cone and of the sphere of radius $2R$ is obviously embedded in the cylinder of axis $\bar x_1 - \bar x_2$ and radius $6R\bar a/\varepsilon_0$, which proves the first result.

• Similarly assume that there exists $\tau^* \geq \delta$ such that
$$|(x_1 - v_1\tau) - (x_2 - v_2\tau)| \leq \varepsilon_0\,.$$
Then, by the triangular inequality again,
$$|(\bar x_1 - \bar x_2) - \tau_*(v_1 - v_2)| \leq \varepsilon_0 + 2\bar a \leq 3\varepsilon_0\,.$$
In particular, for any unit vector n orthogonal to $\bar x_0 - \bar x$,
$$\tau^*|n\cdot(v_1 - v_2)| = |n\cdot((\bar x_1 - \bar x_2) - \tau_*(v_1 - v_2))\,| \leq 3\varepsilon_0\,.$$
This tells us exactly that $v_1 - v_2$ belongs to the cylinder of axis $\bar x_1 - \bar x_2$ and radius $3\varepsilon_0/\delta$.

The lemma is proved. □

The second geometrical lemma requires the use of notation coming from scattering theory, introduced in Chapter 3: it states that if two points z_1, z_2 in $\mathbf{R}^{2d}$ are in a post-collisional configuration and if v_1 or v_2 belong to a cylinder as in Lemma 9.1.3, then the pre-image z_2^* of z_2 through the scattering operator belongs to a small set of $\mathbf{R}^{2d}$.

Lemma 9.1.4. — *Consider two parameters $\rho \ll R$ and $\eta \ll 1$, and $(y,w) \in \mathbf{R}^d \times B_R$. For any v_1 in B_R, define*
$$\mathcal{N}^*(y,w,\rho)(v_1) := \big\{(\nu, v_2) \in \mathbf{S}_1^{d-1} \times B_R \,/\, (v_2 - v_1)\cdot\nu > \eta\,,$$
$$v_1^* \in w + K(y,\rho) \text{ or } v_2^* \in w + K(y,\rho)\big\}\,,$$

where $(\nu^*, v_1^*, v_2^*) = \sigma_0^{-1}(\nu, v_1, v_2)$ *with the notations of Chapter* 3. *Then*

$$|\mathcal{N}^*(y, w, \rho)(v_1)| \leq C(\Phi, R, \eta)\rho^{d-1}$$

where the constant depends on Φ *through the* L^∞ *norm of the cross-section* b *defined in Chapter* 3.

Proof. — Denote by $r = |v_1 - v_2| = |v_1^* - v_2^*|$, and by ω the deflection angle. The formula (3.2.2) shows that, as ω varies in $\mathbf{S}_1^{d-1}$, v_1^* and v_2^* range over a sphere of diameter r.

The solid angle of the intersection of such a sphere with the cylinder $w + K(y, \rho)$ is less than

$$C_d \min\left(1, \left(\frac{\rho}{r}\right)^{d-1}\right)$$

which implies that

$$\begin{aligned} |\{(\omega, v_2) \,/\, v_1^* \in w + K(y,\rho) \text{ or } v_2^* \in w + K(y,\rho)\}| &\leq C_d \int r^{d-1} \min\left(1, \left(\frac{\rho}{r}\right)^{d-1}\right) dr \\ &\leq C_d R \rho^{d-1} \end{aligned}$$

According to Chapter 3, the change of variables $(\nu, v_1 - v_2) \mapsto (\omega, v_1 - v_2)$ is a Lipschitz diffeomorphism away from $\nu \cdot (v_1 - v_2) = 0$. We therefore get the expected estimate. □

Remark 9.1.5. — *Note that those two lemmas consist in eliminating sets in the velocity variables and deflection angles only, and do not concern the position variables.*

9.1.3. Proof of Proposition 9.1.1. — We fix a good configuration $\overline{Z}_k \in \mathcal{G}_k(\varepsilon_0)$, and we consider a configuration $Z_k \in \mathbf{R}^{2dk}$, with the same velocities as $\overline{Z}_k$, and neighboring positions: $|X_k - \overline{X}_k| \leq a$. In particular we notice that for all $\tau \geq 0$ and all $i \neq j$,

$$|x_i - x_j - \tau(\bar v_i - \bar v_j)| \geq |\bar x_i - \bar x_j - \tau(\bar v_i - \bar v_j)| - 2a \geq \varepsilon_0/2 \tag{9.1.7}$$

since $a \ll \varepsilon_0$. This implies that $Z_k \in \mathcal{G}_k(\varepsilon_0/2)$. Next we consider an additional particle $(x_k + \varepsilon\nu, v_{k+1})$ and we shall separate the analysis into two parts, depending on whether the situation is pre-collisional (meaning $\nu \cdot (v_{k+1} - \bar v_k) < 0$) or post-collisional (meaning $\nu \cdot (v_{k+1} - \bar v_k) > 0$).

9.1.3.1. The pre-collisional case. — We assume that

$$\nu \cdot (v_{k+1} - \bar v_k) < 0\,,$$

meaning that $(x_k + \varepsilon\nu, v)$ and z_k form a pre-collisional pair. In particular we have for all times $\tau \geq 0$ and all $\varepsilon > 0$

$$\big|\big(x_k + \varepsilon\nu - v_{k+1}\tau\big) - \big(x_k - \bar v_k \tau\big)\big| \geq \varepsilon\,.$$

Furthermore up to excluding the ball $B_\eta(\bar v_k)$ in the set of admissible v_{k+1}, we may assume that

$$|v_{k+1} - \bar v_k| > \eta\,.$$

Under that assumption we have for all $\tau \geq \delta$ and all $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} \big|\big(x_k + \varepsilon\nu - v_{k+1}\tau\big) - \big(x_k - \bar v_k \tau\big)\big| &\geq \tau|v_{k+1} - \bar v_k| - \varepsilon \\ &\geq \delta\eta - \varepsilon > \varepsilon_0/2\,. \end{aligned}$$

Furthermore we know that Z_k belongs to $\mathcal{G}_k(\varepsilon_0/2)$ thanks to (9.1.7).

Now let $j \in [1, k-1]$ be given. According to Lemma 9.1.3, we find that for any v_{k+1} belonging to the set $B_R \setminus K(\bar x_j - \bar x_k, 6Ra/\varepsilon_0 + 6R\varepsilon_0/\delta)$, we have

$$\forall \tau \geq 0\,, \quad |(x_k + \varepsilon\nu - v_{k+1}\tau) - (x_j - \bar v_j \tau)| > \varepsilon\,,$$

and

$$\forall \tau \geq \delta\,, \quad |(x_k + \varepsilon\nu - v_{k+1}\tau) - (x_j - \bar v_j \tau)| > \varepsilon_0\,.$$

Notice that

$$\Big|B_R \cap K(\bar x_j - \bar x_k, 6Ra/\varepsilon_0 + 6R\varepsilon_0/\delta)\Big| \leq C(R)\left(\frac{a}{\varepsilon_0} + \frac{\varepsilon_0}{\delta}\right)^{d-1}\,.$$

Defining $\mathcal{M}^-(\overline{Z}_k) := \displaystyle\bigcup_{j\leq k-1} K(\bar x_j - \bar x_k, 6Ra/\varepsilon_0 + 6R\varepsilon_0/\delta)$ and

$$\mathcal{B}_k^-(\overline{Z}_k) := \mathbf{S}_1^{d-1} \times \Big(B_\eta(\bar v_k) \cup \mathcal{M}^-(\overline{Z}_k)\Big)$$

we find that

$$\Big|\mathcal{B}_k^-(\overline{Z}_k)\Big| \leq C(R)\Big(\eta^d + \big(\frac{a}{\varepsilon_0}\big)^{d-1} + \big(\frac{\varepsilon_0}{\delta}\big)^{d-1}\Big)$$

and (9.1.3) and (9.1.4) hold as soon as $(\nu, v_{k+1}) \notin \mathcal{B}_k^-(\overline{Z}_k)$.

9.1.3.2. The post-collisional case. — We now assume that

$$\nu \cdot (v_{k+1} - \bar v_k) > 0\,.$$

Next let us define

$$C(\overline{Z}_k) := \Big\{(\nu, v_{k+1}) \in \mathbf{S}_1^d \times B_R\,,\ \nu\cdot(v_{k+1} - \bar v_k) \leq \eta\Big\}\,, \tag{9.1.8}$$

which satisfies

$$|C(\overline{Z}_k)| \leq C(R)\eta^{d-1}\,.$$

Choosing $(\nu, v_{k+1}) \in (\mathbf{S}_1^d \times B_R)\setminus C(\overline{Z}_k)$ ensures that the cross-section is well defined (see Definition 3.3.3), and that the scattering time t_ε is of order $C(R,\eta)\varepsilon$ by Proposition 3.2.1.

Considering the formulas (3.2.2) expressing $(z_k^{\varepsilon*}, z_{k+1}^{\varepsilon*})$ in terms of $\big(z_k, (x_k + \varepsilon\nu, v_{k+1})\big)$, we know that

$$|x_k - x_k^{\varepsilon*}| \leq \varepsilon + Rt_\varepsilon \leq C(R,\eta)\varepsilon \quad \text{and} \quad |(x_k + \varepsilon\nu) - x_{k+1}^{\varepsilon*}| \leq \varepsilon + Rt_\varepsilon \leq C(R,\eta)\varepsilon\,. \tag{9.1.9}$$

Note that due to (9.1.7), all particles x_j with $j \leq k-1$ are at a distance at least $\varepsilon_0/2 - \varepsilon \gg \varepsilon$ of the particles x_k and $x_k + \varepsilon\nu$. Since they have bounded velocities, they cannot enter the protection spheres of these post-collisional particles during the interaction time t_ε, provided that ε is small enough:

$$Rt_\varepsilon \ll \varepsilon_0.$$

Since the dynamics of the particles $j \leq k-1$ is not affected by the scattering, we get that $Z_{k-1}^{\varepsilon*}$ belongs to $\mathcal{G}_{k-1}(\varepsilon_0/2)$:

$$\forall \tau \geq 0\,, \forall (i,j) \in [1, k-1]^2 \text{ with } i \neq j\,, \quad |x_i^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_i^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon_0/2\,. \tag{9.1.10}$$

The pair $(z_k^{\varepsilon*}, z_{k+1}^{\varepsilon*})$ is a pre-collisional pair by definition, so we know that for all $\tau \geq 0$,

$$|(x_k^{\varepsilon*} - \tau v_k^{\varepsilon*}) - (x_{k+1}^{\varepsilon*} - \tau v_{k+1}^{\varepsilon*})| \geq \varepsilon\,.$$

Moreover we have $|v_k^{\varepsilon*} - v^{\varepsilon*}| = |\bar v_k - v_{k+1}| > \eta$, so as in the pre-collisional case above we have

$$\forall \tau \geq \delta\,, \quad |x_k^{\varepsilon*} - x^{\varepsilon*} - \tau(v_k^{\varepsilon*} - v_{k+1}^{\varepsilon*})| \geq \eta\delta - \varepsilon \geq \varepsilon_0\,,$$

for ε sufficiently small, since $\varepsilon_0 \ll \eta\delta$.

Next for $j \leq k-1$ we have for ε sufficiently small, recalling that the uniform, rectilinear motion of the center of mass as described in (3.1.3),

$$|x_j^{\varepsilon*} - \bar{x}_j| \leq |x_j^{\varepsilon*} - x_j| + |x_j - \bar{x}_j| \leq Rt_\varepsilon + a \leq 2a$$

$$\begin{aligned}|x_k^{\varepsilon*} - \bar{x}_k| &\leq \frac{1}{2}|x_k^{\varepsilon*} - x_{k+1}^{\varepsilon^*}| + \frac{1}{2}|(x_k^{\varepsilon*} + x_{k+1}^{\varepsilon^*}) - (x_k + x_{k+1})| + \frac{1}{2}|(x_k + x_{k+1}) - 2\bar{x}_k| \\ &\leq Rt_\varepsilon + \varepsilon + a \leq 2a \\ |x_{k+1}^{\varepsilon*} - \bar{x}_k| &\leq \frac{1}{2}|x_k^{\varepsilon*} - x_{k+1}^{\varepsilon^*}| + \frac{1}{2}|(x_k^{\varepsilon*} + x_{k+1}^{\varepsilon^*}) - (x_k + x_{k+1})| + \frac{1}{2}|(x_k + x_{k+1}) - 2\bar{x}_k| \\ &\leq Rt_\varepsilon + \varepsilon + a \leq 2a\,.\end{aligned}$$

By Lemma 9.1.3, provided $v_k^{\varepsilon*}$ and $v_{k+1}^{\varepsilon*}$ do not belong to

$$\bar{v}_j + K(\bar{x}_j - \bar{x}_k, 12Ra/\varepsilon_0 + 12R\varepsilon_0/\delta) \cap B_R\,,$$

we get since $v_j^{\varepsilon*} = \bar{v}_j$,

$$\begin{aligned}\forall \tau \geq 0\,, \quad &|x_k^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_k^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon\,, \\ \text{and} \quad &|x^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_{k+1}^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon\end{aligned}$$

as well as

$$\begin{aligned}\forall \tau \geq \delta/2\,, \quad &|x_k^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_k^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon_0/2\,, \\ \text{and} \quad &|x^{\varepsilon*} - x_j^{\varepsilon*} - \tau(v_{k+1}^{\varepsilon*} - v_j^{\varepsilon*})| \geq \varepsilon_0/2\,.\end{aligned}$$

$$\begin{aligned}\mathcal{N}^*(\bar{x}_j - \bar{x}_k, \bar{v}_j, \rho)(v_1) := \big\{(\nu, v_2) \in \mathbf{S}_1^{d-1} \times B_R \,/\, (v_2 - v_1)\cdot\nu > \eta\,, \\ v_1^* \in w + K(y,\rho) \text{ or } v_2^* \in w + K(y,\rho)\big\}\end{aligned}$$

Lemma 9.1.4 bounds from the above the size of the set $\mathcal{N}^*((\bar{x}_j - \bar{x}_k, \bar{v}_j, \rho)$ of all $(\nu, v_{k+1}) \in (\mathbf{S}_1^d \times B_R) \setminus C(\overline{Z}_k)$ such that $v_k^{\varepsilon*}$ or $v_{k+1}^{\varepsilon*}$ belongs to $\bar{v}_j + K(\bar{x}_j - \bar{x}_k, \rho)$. We let $\rho = 12Ra/\varepsilon_0 + 12R\varepsilon_0/\delta$, and define

$$\mathcal{M}^+(\overline{Z}_k) := \bigcup_{j \leq k-1} \mathcal{N}^*(\bar{x}_j - \bar{x}_k, \bar{v}_j, \rho)$$

and

$$\mathcal{B}_k^+(\overline{Z}_k) := C(\overline{Z}_k) \cup \big(\mathbf{S}_1^{d-1} \times \mathcal{M}^+(\overline{Z}_k)\big)\,,$$

where the set $C(\overline{Z}_k)$ is introduced in (9.1.8). By Lemma 9.1.4,

$$\Big|\mathcal{B}_k^+(\overline{Z}_k)\Big| \leq C(R)\eta^{d-1} + C(\Phi,\eta,R)\big(\frac{a}{\varepsilon_0} + \frac{\varepsilon_0}{\delta}\big)^{d-1}$$

and (9.1.5) and (9.1.6) hold as soon as $(\nu, v) \notin \mathcal{B}_k^+(\overline{Z}_k)$. The proposition is proved. □

Note that, in order to prove that pathological sets have vanishing measure as $\varepsilon \to 0$, we have to choose η small enough, and then a and ε_0 even smaller in order that (9.1.1) is satisfied and that (9.1.2) is small. Note that, if we want to get a rate of convergence, we need to have more precise bounds on the cross-section b in terms of the truncation parameters R and η.

9.2. Truncation of the Boltzmann and BBGKY pseudo-trajectories

In this paragraph we show that the previous geometrical result may be used to define approximate Boltzmann and BBGKY observables, corresponding to non pathological pseudo-trajectories. We then expect to be able to compare these approximate observables, which will be done in the next chapter.

9.2.1. Initialization. — The first step consists in preparing the initial configuration Z_s so that it is a good configuration. We define

$$\Delta_s(\varepsilon_0) := \Big\{ Z_s \in \mathbf{R}^{ds} \times B_R^s \,/\, \inf_{1\leq \ell < j \leq s} |x_\ell - x_j| \geq \varepsilon_0 \Big\}\,,$$

and we shall assume from now on that Z_s belongs to $\Delta_s(\varepsilon_0)$. We also define for convenience

$$\Delta_s^X(\varepsilon_0) := \Big\{ X_s \in \mathbf{R}^{ds} \,/\, \inf_{1\leq \ell < j \leq s} |x_\ell - x_j| \geq \varepsilon_0 \Big\}\,.$$

Proposition 9.2.1. — *For all $X_s \in \Delta_s^X(\varepsilon_0)$, there is a subset $\mathcal{M}_s(X_s)$ of $\mathbf{R}^{ds}$ such that*

$$\big|\mathcal{M}_s(X_s)\big| \leq C(R,s) \left(\Big(\frac{\varepsilon}{\varepsilon_0}\Big)^{d-1} + \Big(\frac{\varepsilon_0}{\delta}\Big)^{d-1} \right),$$

and defining $\mathcal{P}_s := \Big\{ Z_s \in \Delta_s(\varepsilon_0) \,/\, V_s \in \mathcal{M}_s(X_s) \Big\}$, then

$$\begin{aligned} &\forall \tau \geq 0\,, \quad 1\!\!1_{\mathcal{P}_s} \circ \mathbf{H}_s(\tau) \equiv 1\!\!1_{\mathcal{P}_s} \circ \mathbf{S}_s(\tau) \\ &\forall \tau \geq \delta\,, \quad 1\!\!1_{\mathcal{P}_s} \circ \mathbf{S}_s(\tau) \equiv 1\!\!1_{\mathcal{P}_s} \circ \mathbf{S}_s(\tau) \circ 1\!\!1_{\mathcal{G}_s(\varepsilon_0)}\,. \end{aligned} \tag{9.2.11}$$

denoting abusively by $1\!\!1_A$ the operator of multiplication by the indicator of A.

Proof. — The proof is very similar to the arguments of the previous paragraph. For any Z_s in $\Delta_s(\varepsilon_0)$, we apply Lemma 9.1.3 which shows that outside a small measure set $\mathcal{M}_s(X_s) \subset \mathbf{R}^{ds}$ of velocities $(v_1, \dots, v_s)$, with

$$|\mathcal{M}_s(X_s)| \leq C(R) C_s^2 \left(\frac{\varepsilon}{\varepsilon_0} + \frac{\varepsilon_0}{\delta} \right)^{d-1}\,,$$

the backward nonlinear flow is actually the free flow and the particles remain at a distance larger than ε to one another for all times:

$$\forall \tau > 0, \quad \forall \ell \neq \ell' \in \{1, \dots, s\}\,, \quad |(x_\ell - v_\ell \tau) - (x_{\ell'} - v_{\ell'} \tau)| > \varepsilon\,,$$

and that

$$\forall \tau \geq \delta, \quad \forall \ell \neq \ell' \in \{1, \dots, s\}\,, \quad |(x_\ell - v_\ell \tau) - (x_{\ell'} - v_{\ell'} \tau)| \geq \varepsilon_0\,.$$

By construction, $\mathcal{M}_s(X_s)$ depends continuously on X_s. Therefore, defining $\mathcal{P}_s := \{Z_s \in \Delta_s(\varepsilon_0) \,/\, V_s \notin \mathcal{M}_s(X_s)\}$ gives the result. □

9.2.2. Approximation of the Boltzmann functional. — We recall that we consider a family of initial data $F_0 = (f_0^{(s)})$ satisfying

$$|||F_0|||_{0,\beta_0,\mu_0} = \sup_{s\in\mathbf{N}} \sup_{Z_s} \big(\exp(\beta_0 E(Z_s) + \mu_0 s) f_0^{(s)}(Z_s) \big) < +\infty$$

and after the reductions of Chapter 8, the observable we are interested in is the following:

$$\begin{aligned} I_{s,n}^{0,R,\delta}(t,J,M)(X_s) := \int \varphi_s(V_s) \int_{\mathcal{T}_{n,\delta}(t)} &\mathbf{S}_s(t-t_1) \mathcal{C}_{s,s+1}^{0,j_1,m_1} \mathbf{S}_{s+1}(t_1-t_2) \mathcal{C}_{s+1,s+2}^{0,j_2,m_2} \\ &\dots \mathbf{S}_{s+n}(t_n - t_{n+1}) 1\!\!1_{|E_0(Z_{s+n})| \leq R^2} f_0^{(s+n)} dT_n dV_s\,, \end{aligned} \tag{9.2.12}$$

By Proposition 9.2.1, up to an error term of order $C(R)s^2\,(\varepsilon/\varepsilon_0+\varepsilon_0/\delta)^{d-1}$, we can assume that the initial configuration Z_s is a good configuration, meaning that

$$I^{0,R,\delta}_{s,n}(t,J,M)(X_s)=\int_{B_R\setminus\mathcal{M}_s(X_s)}\varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)}\mathbf{S}_s(t-t_1)\mathcal{C}^{0,j_1,m_1}_{s,s+1}\mathbf{S}_{s+1}(t_1-t_2)\mathcal{C}^{0,j_2,m_2}_{s+1,s+2}$$
$$\dots\mathcal{C}^{0,j_n,m_n}_{s+n-1,s+n}\mathbf{S}_{s+n}(t_n-t_{n+1})1\!\!1_{|E_0(Z_{s+n})|\leq R^2}f_0^{(s+n)}\,dT_n dV_s$$
$$+O\Big(C(R)s^2\left(\frac{\varepsilon}{\varepsilon_0}+\frac{\varepsilon_0}{\delta}\right)^{d-1}\Big)\,,$$

where

$$\big(\mathcal{C}^{0,-,m}_{s,s+1}f^{(s+1)}\big)(Z_s)=\int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d}((v_{s+1}-v_m)\cdot\nu_{s+1})_-f^{(s+1)}(Z_s,x_m,v_{s+1})\,d\nu_{s+1}dv_{s+1}\quad\text{and}$$
$$\big(\mathcal{C}^{0,+,m}_{s,s+1}f^{(s+1)}\big)(Z_s)=\int_{\mathbf{S}_1^{d-1}\times\mathbf{R}^d}((v_{s+1}-v_m)\cdot\nu_{s+1})_+f^{(s+1)}(z_1,\dots,x_m,v^*_m,\dots,z_s,x_m,v^*_{s+1})\,d\nu_{s+1}dv_{s+1}\,.$$

Now let us introduce some notation which we shall be using constantly from now on: given $Z_s\in\Delta_s(\varepsilon_0)$, we call $Z^0_s(\tau)$ the position of the backward free flow initiated from Z_s, at time $t_1\leq\tau\leq t$. Then given $j_1\in\{+,-\}$, $m_1\in[1,s]$, a deflection angle ω_{s+1} and a velocity v_{s+1} we call $Z^0_{s+1}(\tau)$ the position at time $t_2\leq\tau<t_1$ of the Boltzmann pseudo-trajectory initiated by the adjunction of the particle (ν_{s+1},v_{s+1}) to the particle $z^0_{m_1}(t_1)$ (which is simply free-flow in the pre-collisional case $j_1=-$, and free-flow after scattering of particles $z^0_{m_1}(t_1)$ and (ν_{s+1},v_{s+1}) in the post-collisional case $j_1=+$).

Similarly by induction given $Z_s\in\Delta_s(\varepsilon_0)$, T,J and M we denote for each $1\leq k\leq n$ by $Z^0_{s+k}(\tau)$ the position at time $t_{k+1}\leq\tau<t_k$ of the pseudo-trajectory initiated by the adjunction of the particle (ν_{s+k},v_{s+k}) to the particle $z^0_{m_k}(t_k)$ (which is simply free-flow in the pre-collisional case $j_k=-$, and free-flow after scattering of particles $z^0_{m_k}(t_k)$ and (ν_{s+k},v_{s+k}) in the post-collisional case $j_k=+$).

Notice that $\tau\mapsto\ Z^0_{s+k}(\tau)$ is pointwise right-continuous on $[0,t_k]$.

With this notation, the functional $I^{0,R,\delta}_{s,n}$ may be reformulated as

$$I^{0,R,\delta}_{s,n}(t,J,M)(X_s)=\int_{B_R\setminus\mathcal{M}_s(X_s)}dV_s\varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)}dT_n\int_{\mathbf{S}_1^{d-1}\times B_R}d\nu_{s+1}dv_{s+1}((v_{s+1}-v^0_{m_1}(t_1)\cdot\nu_{s+1})_+$$
$$\dots\int_{\mathbf{S}_1^{d-1}\times B_R}d\nu_{s+n}dv_{s+n}((v_{s+n}-v^0_{m_n}(t_n)\cdot\nu_{s+n})_+1\!\!1_{E_0(Z^0_{s+n}(0))\leq R^2}f_0^{(s+n)}(Z^0_{s+n}(0))$$
$$+\,O\Big(C(R)s^2\left(\frac{\varepsilon}{\varepsilon_0}+\frac{\varepsilon_0}{\delta}\right)^{d-1}\Big)\,.$$

Let $a,\varepsilon_0,\eta\ll1$ be such that

$$a\ll\varepsilon_0\ll\eta\delta\,.$$

According to Proposition 9.1.1, for any good configuration $\overline{Z}_{s+k-1}\in\mathbf{R}^{2d(s+k-1)}$, we can define a set

$${}^c\mathcal{B}_{s+k-1}(\overline{Z}_{s+k-1}):=\big(\mathbf{S}_1^{d-1}\times B_R\big)\setminus\mathcal{B}_{s+k-1}(\overline{Z}_{s+k-1})\,,$$

such that good configurations $Z_{s+k-1}=(X_{s+k-1},\overline{V}_{s+k-1})$ with $|X_{s+k-1}-\overline{X}_{s+k-1}|\leq Ca$ are stable by adjunction of a collisional particle $(\nu_{k+s},v_{k+s})\in^c\mathcal{B}^{(}_{s+k-1}\overline{Z}_{s+k-1})$.

We further notice that thanks to Remark 9.1.2, if the adjoined pair (ν_{s+k},v_{s+k}) belongs to the set ${}^c\mathcal{B}_{s+k-1}(Z^0_{s+k-1}(t_k))$ with $Z^0_{s+k-1}(t_k)\in\mathcal{G}_{s+k-1}(\varepsilon_0)$, then $Z^0_{s+k}(t_{k+1})$ belongs to $\mathcal{G}_{s+k}(\varepsilon_0)$.

As a consequence we may define recursively the approximate Boltzmann functional

$$
\begin{aligned}
J^{0,R,\delta}_{s,n}(t,J,M)(X_s) = \int_{B_R\setminus\mathcal{M}_s(X_s)} & dV_s\varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)} dT_n \\
& \int_{^c\mathcal{B}_s(Z^0_s(t_1))} d\nu_{s+1}dv_{s+1}(v_{s+1}-v^0_{m_1}(t_1)\cdot\nu_{s+1})_{j_1} \\
& \quad \dots \int_{^c\mathcal{B}_{s+n-1}i(Z^0_{s+n-1}(t_n))} d\nu_{s+n}dv_{s+n}(v_{s+n}-v^0_{m_n}(t_n)\cdot\nu_{s+n})_{j_n} \\
& \qquad \times 1\!\!1_{E_0(Z^0_{s+n}(0))\leq R^2} f_0^{(s+n)}(Z^0_{s+n}(0))\,.
\end{aligned}
\tag{9.2.13}
$$

The following result is an immediate consequence of Proposition 9.1.1

Proposition 9.2.2. — *Let $a,\varepsilon_0,\eta\ll 1$ satisfying* (9.1.1)*. Then,*

$$
\Big|1\!\!1_{\Delta_s(\varepsilon_0)}\big(I^{0,R,\delta}_{s,n}-J^{0,R,\delta}_{s,n}\big)(t,J,M)\Big| \leq (s+n)^2\left(C(R)\eta^{d-1}+C(R,\Phi,\eta)\Big(\frac{a}{\varepsilon_0}+\frac{\varepsilon_0}{\delta}\Big)^{d-1}\right)|\!|\!|F_0|\!|\!|_{0,\beta_0,\mu_0}\,.
$$

9.2.3. Approximation of the BBGKY functional. — We recall that after the reductions of Chapter 8, the functional we are interested in is

$$
\begin{aligned}
I^{R,\delta}_{s,n}(t,J,M)(X_s) := \int\varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)} & \mathbf{H}_s(t-t_1)\mathcal{C}^{j_1,m_1}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\mathcal{C}^{j_2,m_2}_{s+1,s+2} \\
& \dots\mathcal{C}^{j_n,m_n}_{s+n-1,s+n}\mathbf{H}_{s+n}(t_n-t_{n+1})1\!\!1_{|E_\varepsilon(Z_{s+n})|\leq R^2}\widetilde{f}^{(s+n)}_{N,0}dT_ndV_s\,,
\end{aligned}
$$

where $\widetilde{F}_{N,0}=(\widetilde{f}^{(s)}_{N,0})_{1\leq s\leq N}$ satisfies

$$
|\!|\!|\widetilde{F}_{N,0}|\!|\!|_{\varepsilon,\beta_0,\mu_0} = \sup_{s\in\mathbf{N}}\sup_{Z_s}\big(\exp(\beta_0E_\varepsilon(Z_s)+\mu_0 s)\widetilde{f}^{(s)}_{N,0}(Z_s)\big) < +\infty\,.
$$

Thanks to Proposition 9.2.1, we have

$$
\begin{aligned}
I^{R,\delta}_{s,n}(t,J,M)(X_s) = \int_{B_R\setminus\mathcal{M}_s(X_s)}\varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)} & \mathbf{S}_s(t-t_1)1\!\!1_{\mathcal{G}_s(\varepsilon_0)}\mathcal{C}^{j_1,m_1}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2)\mathcal{C}^{j_2,m_2}_{s+1,s+2} \\
& \dots\mathcal{C}^{j_n,m_n}_{s+n-1,s+n}\mathbf{H}_{s+n}(t_n-t_{n+1})1\!\!1_{E_\varepsilon(Z_{s+n}(0))\leq R^2}\widetilde{f}^{(s+n)}_{N,0}dT_ndV_s \\
& +O\Big(C(R)s^2\left(\frac{\varepsilon}{\varepsilon_0}+\frac{\varepsilon_0}{\delta}\right)^{d-1}\Big)\,.
\end{aligned}
$$

Then using the notation introduced in the previous paragraph for the Boltzmann pseudo-trajectory, let us define the approximate functionals

$$
\begin{aligned}
J^{R,\delta}_{s,n}(t,J,M)(X_s) := \int_{B_R\setminus\mathcal{M}_s(X_s)}\varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)} & \mathbf{S}_s(t-t_1)1\!\!1_{\mathcal{G}_s(\varepsilon_0)}\widetilde{\mathcal{C}}^{j_1,m_1}_{s,s+1}\mathbf{H}_{s+1}(t_1-t_2) \\
& \dots\widetilde{\mathcal{C}}^{j_n,m_n}_{s+n-1,s+n}\mathbf{H}_{s+n}(t_n-t_{n+1})1\!\!1_{E_\varepsilon(Z_{s+n}(0))\leq R^2}\widetilde{f}^{(s+n)}_0dT_ndV_s\,,
\end{aligned}
$$

where

$$\begin{aligned}\big(\widetilde{\mathcal{C}}^{-,m_k}_{s+k-1,s+k}g^{(s+k)}\big)(Z_{s+k-1}) &:= (N-s-k+1)\varepsilon^{d-1}\int_{{}^c\mathcal{B}_{s+k-1}(Z^0_{s+k-1}(t_k))}(\nu_{s+k}\cdot(v_{s+k}-v_{m_k}(t_k)))_-\\ &\times g^{(s+k)}(\cdot,x_{m_k}(t_k)+\varepsilon\nu_{s+k},v_{s+k}(t_k))\prod_{\substack{1\leq j\leq s+k-1\\ j\neq m_k}}1\!\!1_{|(x_j-x_{m_k})(t_k)-\varepsilon\nu_{s+k}|\geq\varepsilon}\,d\nu_{s+k}dv_{s+k}\\ \big(\widetilde{\mathcal{C}}^{+,m_k}_{s+k-1,s+k}g^{(s+k)}\big)(Z_{s+k-1}) &:= (N-s-k+1)\varepsilon^{d-1}\int_{{}^c\mathcal{B}_{s+k-1}(Z^0_{s+k-1}(t_k))}(\nu_{s+k}\cdot(v_{s+k}-v_{m_k}(t_k)))_+\\ &\times g^{(s+k)}(\dots,x^*_{m_k},v^*_{m_k},\dots,x^*_{s+k},v^*_{s+k})\prod_{\substack{1\leq j\leq s+k-1\\ j\neq m_k}}1\!\!1_{|(x_j-x_{m_k})(t_k)-\varepsilon\nu_{s+k}|\geq\varepsilon}\,d\nu_{s+k}dv_{s+k}\,.\end{aligned}$$

denoting as previously by $(x^*_{m_k},v^*_{m_k},x^*_{s+k},v^*_{s+k})$ the pre-image of $(x_{m_k},v_{m_k}(t_k),x_{m_k}(t_k)+\varepsilon\nu_{s+k},v_{s+k}(t_k))$ by the scattering σ_ε.

As in the Boltzmann case described above, the following result is an immediate consequence of Proposition 9.1.1.

Proposition 9.2.3. — *Let $a,\varepsilon_0,\eta\ll 1$ satisfying* (9.1.1). *Then, for ε sufficiently small,*

$$\Big|1\!\!1_{\Delta_s(\varepsilon_0)}\big(I^{R,\delta}_{s,n}-J^{R,\delta}_{s,n}\big)(t,J,M)\Big|\leq(s+n)^2\left(C(R)\eta^{d-1}+C(R,\Phi,\eta)\Big(\frac{a}{\varepsilon_0}+\frac{\varepsilon_0}{\delta}\Big)^{d-1}\right)|||\widetilde{F}_{N,0}|||_{\varepsilon,\beta_0,\mu_0}\,.$$

The functional $J^{R,\delta}_{s,n}$ can be written in terms of pseudo-trajectories, as in (9.2.13). Let us therefore introduce some notation which we shall be using constantly from now on: given $Z_s\in\Delta_s(\varepsilon_0)$, we call $Z^0_s(\tau)$ the position of the backward free flow initiated from Z_s, at time $t_1\leq\tau\leq t$. Then given $j_1\in\{+,-\}$, $m_1\in[1,s]$, an angle ν_{s+1} (or equivalently a position $x_{s+1}=x^0_{m_1}(t_1)+\varepsilon\nu_{s+1}$) and a velocity v_{s+1} we call $Z^\varepsilon_{s+1}(\tau)$ the position at time $t_2\leq\tau<t_1$ of the BBGKY pseudo-trajectory initiated by the adjunction of the particle z_{s+1} to the particle $z^0_{m_1}(t_1)$.

Similarly by induction given $Z_s\in\Delta_s(\varepsilon_0)$, T,J and M we denote for each $1\leq k\leq n$ by $Z^\varepsilon_{s+k}(\tau)$ the position at time $t_{k+1}\leq\tau<t_k$ of the BBGKY pseudo-trajectory initiated by the adjunction of the particle z_{s+k} to the particle $z_{m_k}(t_k)$. We have

$$\begin{aligned}J^{R,\delta}_{s,n}(t,J,M)(X_s) &= \frac{(N-s)!}{(N-s-n)!}\varepsilon^{n(d-1)}\int_{B_R\setminus\mathcal{M}_s(X_s)}dV_s\varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)}dT_n\\ &\int_{{}^c\mathcal{B}_s(Z^0_s(t_1))}d\nu_{s+1}dv_{s+1}\,(\nu_{s+1}\cdot(v_{s+1}-v_{m_1}(t_1)))_{j_1}\prod_{\substack{1\leq j\leq s\\ j\neq m_1}}1\!\!1_{|(x_j-x_{m_1})(t_1)-\varepsilon\nu_{s+1}|\geq\varepsilon}\\ &\dots\int_{{}^c\mathcal{B}^{j_n}_{s+n-1}(Z^0_{s+n-1}(t_n))}d\nu_{s+n}dv_{s+n}\,(\nu_{s+n}\cdot(v_{s+n}-v_{m_n}(t_n)))_{j_n}\\ &\times\prod_{\substack{1\leq j\leq s+n-1\\ j\neq m_n}}1\!\!1_{|(x_j-x_{m_n})(t_n)-\varepsilon\nu_{s+n}|\geq\varepsilon}1\!\!1_{E_\varepsilon(Z_{s+n}(0))\leq R^2}\widetilde{f}^{(s+n)}_{N,0}(Z^\varepsilon_{s+n}(0))\,.\end{aligned}\tag{9.2.14}$$

Thanks to Propositions 9.2.2 and 9.2.3 the proof of Theorem 4 reduces to the proof of the convergence to zero of $J^{R,\delta}_{s,n}-J^{0,R,\delta}_{s,n}$. This is the object of the next chapter.

CHAPTER 10

CONVERGENCE PROOF

In this chapter we conclude the proof of Theorem 4 by proving that $J^{R,\delta}_{s,n} - J^{0,R,\delta}_{s,n}$ goes to zero in the Boltzmann-Grad limit, with the notation of the previous chapter, namely (9.2.13) and (9.2.14). The main difficulty lies in the fact that in contrast to the Boltzmann situation, collisions in the BBGKY configuration are not pointwise in space, nor in time. At each collision time t_k a small error is therefore introduced, which needs to be controled.

We recall that, as in the previous chapter, we consider dynamics
– involving only a finite number $s+n$ of particles,
– with bounded energies (at most $R^2 \gg 1$),
– such that the n additional particles are adjoined through binary collisions at times separated at least by $\delta \ll 1$.
The additional truncation parameters $a, \varepsilon_0, \eta \ll 1$ satisfy (9.1.1).

10.1. Proximity of Boltzmann and BBGKY trajectories

This paragraph is devoted to the proof, by induction, that the BBGKY and Boltzmann pseudo-trajectories remain close for all times, in particular that there is no recollision for the BBGKY dynamics.

We recall that the notation $Z^0_k(t)$ and $Z_k(t)$ were defined in Paragraphs 9.2.2 and 9.2.3 respectively.

Lemma 10.1.1. — *Fix $T \in \mathcal{T}_{n,\delta}(t)$, J, and M and given Z_s in $\Delta_s(\varepsilon_0)$, consider for all $i \in \{1, \dots n\}$, an impact parameter ν_{s+i} and a velocity v_{s+i} such that $(\nu_{s+i}, v_{s+i}) \notin \mathcal{B}_{s+i-1}(Z^0_{s+i-1}(t_i))$. Then, for ε sufficiently small, for all $i \in [1, n]$, and all $k \leq s+i$,*

$$|x^\varepsilon_k(t_{i+1}) - x^0_k(t_{i+1})| \leq C\varepsilon i \quad \text{and} \quad v_k(t_{i+1}) = v^0_k(t_{i+1})\,, \tag{10.1.1}$$

where the constant C depends only on η, R, δ.

Proof. — We proceed by induction on i, the index of the time variables t_{i+1} for $0 \leq i \leq n$.

We first notice that by construction, $Z_s(t_1) - Z^0_s(t_1) = 0$, so (10.1.1) holds for $i = 0$. The initial configuration being a good configuration, we indeed know that there is no possible recollision.

Now let $i \in [1, n]$ be fixed, and assume that for all $\ell \leq i$

$$\forall k \leq s+\ell-1, \qquad |x^\varepsilon_k(t_\ell) - x^0_k(t_\ell)| \leq C\varepsilon(\ell-1) \quad \text{and} \quad v_k(t_\ell) = v^0_k(t_\ell)\,. \tag{10.1.2}$$

Let us prove that (10.1.2) holds for $\ell = i+1$. We shall consider two cases depending on whether the particle adjoined at time t_i is pre-collisional or post-collisional.

• As usual, the case of pre-collisional velocities $(v_{s+i}, v_{m_i}(t_i))$ at time t_i is the most simple to handle. We indeed have $\forall \tau \in [t_{i+1}, t_i]$

$$\begin{aligned}&\forall k < s+i\,, \quad x^0_k(\tau) = x^0_k(t_i) + (\tau - t_i) v^0_k(t_i)\,, \qquad v^0_k(\tau) = v^0_k(t_i)\,,\\ &x^0_{s+i}(\tau) = x^0_{m_i}(t_i) + (\tau - t_i) v_{s+i}\,, \qquad v^0_{s+i}(\tau) = v_{s+i}\,.\end{aligned}$$

Now let us study the BBGKY trajectory. We recall that the particle is adjoined in such a way that (ν_{s+i}, v_{s+i}) belongs to ${}^c\mathcal{B}_{s+i-1}(Z^0_{s+i-1}(t_i))$. Provided that ε is sufficiently small, by the induction assumption (10.1.2), we have

$$|X^\varepsilon_{s+i-1}(t_i) - X^0_{s+i-1}(t_i)| \leq C\varepsilon(i-1) \leq a\,.$$

Since $Z^0_{s+i-1}(t_i)$ belongs to $\mathcal{G}_{s+i-1}(\varepsilon_0)$ (see Paragraph 9.2.2), we can apply Proposition 9.1.1 which implies that backwards in time, there is free flow for Z^ε_{s+i}. In particular,

$$\begin{aligned}&\forall k < s+i\,, \quad x_k(\tau) = x_k(t_i) + (\tau - t_i) v_k(t_i)\,, \qquad v_k(\tau) = v_k(t_i)\,,\\ &x_{s+i}(\tau) = x_{m_i}(t_i) + \varepsilon \nu_{s+i} + (\tau - t_i) v_{s+i}\,, \qquad v_{s+i}(\tau) = v_{s+i}\,.\end{aligned}$$

We therefore obtain

$$\forall k \leq s+i\,, \quad \forall \tau \in [t_{i+1}, t_i]\,, \quad v_k(\tau) - v^0_k(\tau) = v_k(t_i) - v^0_k(t_i) = 0\,, \tag{10.1.3}$$

and

$$\forall k \leq s+i\,, \quad \forall \tau \in [t_{i+1}, t_i]\,, \quad |x_k(\tau) - x^0_k(\tau)| \leq C\varepsilon(i-1) + \varepsilon\,. \tag{10.1.4}$$

• The case of post-collisional velocities is a little more complicated since there is a (small) time interval during which interaction occurs.

Let us start by describing the Boltzmann flow. By definition of the post-collisional configuration, we know that the following identifies hold:

$$\forall t_{i+1} \leq \tau < t_i\,, \quad \begin{cases} (v^0_{m_i}, v^0_{s+i})(\tau) = (v^{0*}_{m_i}(t_i), v^*_{s+i}) \text{ with } (\nu^*_{s+i}, v^{0*}_{m_i}(t_i), v^*_{s+i}) := \sigma_0^{-1}(\nu_{s+i}, v^0_{m_i}(t_i), v_{s+i})\\ x^0_{m_i}(\tau) = x^0_{m_i}(t_i) + (\tau - t_i) v^{0*}_{m_i}(t_i)\,,\ x^0_{s+i}(\tau) = x^0_{s+i}(t_i) + (\tau - t_i) v^*_{s+i}\\ \forall j \notin \{m_i, s+1\}\,, \quad v^0_j(\tau) = v^0_j(t_i)\,,\ x^0_j(\tau) = x^0_j(t_i) + (\tau - t_i) v^0_j(t_i)\,, \end{cases}$$

where σ_0 denotes the scattering operator defined in Definition 3.2.1 in Chapter 3.

First, by Proposition 9.1.1, we know that for $j \notin \{m_i, s+i\}$ and $\forall \tau \in [t_{i+1}, t_i]$,

$$x_j(\tau) = x_j(t_i) + (\tau - t_i) v_j(t_i)\,, \qquad v_j(\tau) = v_j(t_i)\,,$$

so that by the induction assumption (10.1.2) we obtain

$$\begin{aligned}\forall j \notin \{m_i, s+i\}\,, \forall \tau \in [t_{i+1}, t_i]\,, \quad |x_j(\tau) - x^0_j(\tau)| &= |x_j(t_i) - x^0_j(t_i)| \leq C\varepsilon\\ &\text{and} \quad v_j(\tau) = v^0_j(\tau)\,.\end{aligned} \tag{10.1.5}$$

We now have to focus on the pair $(s+i, m_i)$. According to Chapter 3, the relative velocity evolves under the nonlinear dynamics on a time interval $[t_i - t_\varepsilon, t_i]$ with $t_\varepsilon \leq C(\eta, R)\varepsilon$ (recalling that by construction, the relative velocity $|v_{s+i} - v_{m_i}(t_i)|$ is bounded from above by R and from below by η, and that the impact parameter is also bounded from below by η). Then, for all $\tau \in [t_{i+1}, t_i - t_\varepsilon]$,

$$v_{s+i}(\tau) = v^*_{s+i} = v^0_{s+i}(\tau)\,, \quad v_{m_i}(\tau) = v^*_{m_i}(t_i) = v^{0*}_{m_i}(t_i) = v^0_{m_i}(\tau)\,. \tag{10.1.6}$$

In particular,

$$v_{s+i}(t_{i+1}) = v^0_{s+i}(t_{i+1}) \quad \text{and} \quad v_{m_i}(t_{i+1}) = v^0_{m_i}(t_{i+1})\,. \tag{10.1.7}$$

The conservation of total momentum as in Paragraph 9.1.3.2 shows that

$$\begin{aligned}\Big|\frac12(x^\varepsilon_{m_i}(t_i-t_\varepsilon)+x^\varepsilon_{s+i}(t_i-t_\varepsilon))-\frac12(x^0_{m_i}(t_i-t_\varepsilon)+x^0_{s+i}(t_i-t_\varepsilon))\Big|&\\ =\Big|\frac12(x^\varepsilon_{m_i}(t_i)+x^\varepsilon_{s+i}(t_i)-\frac12(x^0_{m_i}(t_i)+x^0_{s+i}(t_i))\Big|&\\ =\Big|x^\varepsilon_{s+i}(t_i)-x^0_{s+i}(t_i)\Big|+\frac{\varepsilon}{2}\leq C\varepsilon(i-1)+\frac{\varepsilon}{2}\,\cdot&\end{aligned}$$

On the other hand, by definition of the scattering time t_ε,

$$\begin{aligned}|x^\varepsilon_{m_i}(t_i-t_\varepsilon)-x^\varepsilon_{s+i}(t_i-t_\varepsilon)|&=\varepsilon\,,\\ |x^0_{m_i}(t_i-t_\varepsilon)-x^0_{s+i}(t_i-t_\varepsilon)|&=t_\varepsilon|v^*_{m_i}-v^*_{s+i}|\leq C(\eta,R)\varepsilon\,.\end{aligned}$$

We obtain finally

$$|x^\varepsilon_{m_i}(t_i-t_\varepsilon)-x^0_{m_i}(t_i-t_\varepsilon)|\leq C\varepsilon i \text{ and } |x^\varepsilon_{s+i}(t_i-t_\varepsilon)-x^0_{s+i}(t_i-t_\varepsilon)|\leq C\varepsilon i \tag{10.1.8}$$

provided that C is chosen sufficiently large (depending on R and η).

Now let us apply Proposition 9.1.1, which implies that for all $\tau\in[t_{i+1},t_i-t_\varepsilon]$ the backward in time evolution of the two particles $x^\varepsilon_{s+i}(t_i-t_\varepsilon)$ and $x^\varepsilon_{m_i}(t_i-t_\varepsilon)$, is that of free flow: we have therefore, using (10.1.6),

$$\begin{aligned}x^\varepsilon_{m_i}(t_{i+1})-x^0_{m_i}(t_{i+1})&=x^\varepsilon_{m_i}(t_i-t_\varepsilon)-x^0_{m_i}(t_i-t_\varepsilon)\,,\\ x^\varepsilon_{s+i}(t_{i+1})-x^0_{s+i}(t_{i+1})&=x^\varepsilon_{s+i}(t_i-t_\varepsilon)-x^0_{s+i}(t_i-t_\varepsilon)\,.\end{aligned}$$

From (10.1.8) we therefore deduce that the induction assumption is satisfied at time step t_{i+1}, and the proposition is proved. □

Note that, by construction,

$$Z^0_{s+n}(0)\in\mathcal{G}_{s+n}(\varepsilon_0)\,,$$

so that an obvious application of the triangular inequality leads to

$$Z^\varepsilon_{s+n}(0)\in\mathcal{G}_{s+n}(\varepsilon_0/2)\,.$$

Note also that the indicator functions are identically equal to 1 for good configurations. We therefore have the following

Corollary 10.1.2. — *Under the assumptions of Lemma* 10.1.1*, the functional* $J^{R,\delta}_{s,n}(t,J,M)$ *defined in* (9.2.14) *may be written as follows:*

$$\begin{aligned}J^{R,\delta}_{s,n}(t,J,M)(X_s)=\frac{(N-s)!}{(N-s-n)!}\varepsilon^{n(d-1)}&\int_{B_R\setminus\mathcal{M}_s(X_s)}dV_s\varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)}dT_n\\ &\int_{^c\mathcal{B}_s(Z^0_s(t_1))}d\nu_{s+1}dv_{s+1}\,(\nu_{s+1}\cdot(v_{s+1}-v_{m_1}(t_1)))_{j_1}\\ &\dots\int_{^c\mathcal{B}_{s+n-1}(Z^0_{s+n-1}(t_n))}d\nu_{s+n}dv_{s+n}\,(\nu_{s+n}\cdot(v_{s+n}-v_{m_n}(t_n)))_{j_n}\\ &\times 1\!\!1_{E_\varepsilon(Z_{s+n}(0))\leq R^2}1\!\!1_{Z_{s+n}(0)\in\mathcal{G}_{s+n}(\varepsilon_0/2)}\widetilde{f}^{(s+n)}_{N,0}(Z^\varepsilon_{s+n}(0))\,.\end{aligned}$$

10.2. End of the proof of Theorem 4

The end of the proof of Theorem 4 consists in estimating the error terms in $J^{R,\delta}_{s,n} - J^{0,R,\delta}_{s,n}$ coming essentially from the micro-translations described in the previous paragraph and from the initial data.

10.2.1. Error coming from the initial data. — Let us replace the initial data in $J^{R,\delta}_{s,n}$ by that of the Boltzmann hierarchy, defining:

$$\begin{aligned}\widetilde{J}^{R,\delta}_{s,n}(t,J,M)(X_s) = \frac{(N-s)!}{(N-s-n)!}\varepsilon^{n(d-1)} &\int_{B_R\setminus\mathcal{M}_s(X_s)} dV_s \varphi_s(V_s) \int_{\mathcal{T}_{n,\delta}(t)} dT_n \\ &\int_{{}^c\mathcal{B}_s(Z^0_s(t_1))} d\nu_{s+1} dv_{s+1}\,(\nu_{s+1}\cdot(v_{s+1}-v_{m_1}(t_1)))_{j_1} \\ &\quad \dots \int_{{}^c\mathcal{B}_{s+n-1}(Z^0_{s+n-1}(t_n))} d\nu_{s+n} dv_{s+n}\,(\nu_{s+n}\cdot(v_{s+n}-v_{m_n}(t_n)))_{j_n} \\ &\quad \times 1\!\!1_{E_0(Z_{s+n}(0))\le R^2} 1\!\!1_{Z^\varepsilon_{s+n}(0)\in\mathcal{G}_{s+n}(\varepsilon_0/2)} f_0^{(s+n)}(Z_{s+n}(0))\,.\end{aligned}$$

Lemma 10.2.1. — *In the Boltzmann-Grad scaling $N\varepsilon^{d-1}=1$,*

$$\big|1\!\!1_{\Delta^X_s(\varepsilon_0)}(J^{R,\delta}_{s,n}-\widetilde{J}^{R,\delta}_{s,n})(t,J,M)(X_s)\big| \le C\frac{R^{dn}T^n}{n!}\|\varphi_s\|_{L^\infty(\mathbf{R}^{ds})}\big\|1\!\!1_{\Delta_{s+n}(\varepsilon_0/2)}(\widetilde{f}^{(s+n)}_{N,0}-f_0^{(s+n)})\big\|_{L^\infty(\mathbf{R}^{2d(s+n)})}$$

and in particular

$$|1\!\!1_{\Delta^X_s(\varepsilon_0)}(\widetilde{I}_{s,n}-\widetilde{J}_{s,n})(t,J,M)(X_s)| \to 0 \ \text{as}\ \varepsilon\to 0\,,$$

uniformly in $t\in[0,T]$ and $X_s\in\mathbf{R}^{ds}$.

Proof. — We recall that by assumption, $1\!\!1_{\Delta_{s+n}(\varepsilon_0/2)}(\widetilde{f}^{(s+n)}_{N,0} - f_0^{(s+n)})$ goes to zero uniformy in Z_{s+n}.

By definition of the good sets $\mathcal{G}_k(c)$, the positions in the argument of $\widetilde{f}^{(s+n)}_{N,0} - f_0^{(s+n)}$ satisfy the separation condition $|x_i - x_j| \ge \varepsilon_0/2$ for $i\neq j$:

$$1\!\!1_{\mathcal{G}_{s+n}(\varepsilon_0/2)}(\widetilde{f}^{(s+n)}_{N,0} - f_0^{(s+n)}) = 1\!\!1_{\mathcal{G}_{s+n}(\varepsilon_0/2)}1\!\!1_{\Delta_{s+n}(\varepsilon_0/2)}(\widetilde{f}^{(s+n)}_{N,0} - f_0^{(s+n)})\,.$$

Furthermore, due to that separation condition,

$$E_\varepsilon(Z_{s+n}) = E_0(Z_{s+n}) = \frac12\sum_{i=1}^{s+n}|v_i|^2\,.$$

So we can write

$$\begin{aligned}(J^{R,\delta}_{s,n}(t,J,M) - \widetilde{J}^{R,\delta}_{s,n}(t,J,M))(X_s) = \frac{(N-s)!}{(N-s-n)!}\varepsilon^{n(d-1)} &\int_{B_R\setminus\mathcal{M}_s(X_s)} dV_s\varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)} dT_n \\ &\int_{{}^c\mathcal{B}_s(Z^0_s(t_1))} d\nu_{s+1}dv_{s+1}(\nu_{s+1}\cdot(v_{s+1}-v_{m_1}(t_1)))_{j_1} \\ &\quad\dots\int_{{}^c\mathcal{B}_{s+n-1}(Z^0_{s+n-1}(t_n))} d\nu_{s+n}dv_{s+n}\,(\nu_{s+n}\cdot(v_{s+n}-v_{m_n}(t_n)))_{j_n} \\ &\quad\times 1\!\!1_{E_\varepsilon(Z^\varepsilon_{s+n}(0))\le R^2}1\!\!1_{\Delta_{s+n}(\varepsilon_0/2)}(\widetilde{f}^{(s+n)}_{N,0} - f_0^{(s+n)})\,,\end{aligned}$$

and we find directly that

$$\begin{aligned}&\Big|1\!\!1_{\Delta^X_s(\varepsilon_0)}(J^{R,\delta}_{s,n}(t,J,M) - \widetilde{J}^{R,\delta}_{s,n}(t,J,M))(X_s)\Big| \\ &\quad\le C\frac{R^{dn}T^n}{n!}\|\varphi_s\|_{L^\infty(\mathbf{R}^{ds})}\big\|1\!\!1_{\Delta_{s+n}(\varepsilon_0/2)}(\widetilde{f}^{(s+n)}_{N,0} - f_0^{(s+n)})\big\|_{L^\infty(\mathbf{R}^{2d(s+n)})}\,.\end{aligned}$$

The result is proved. □

10.2.2. Error coming from the prefactors in the collision operators. — As $\varepsilon \to 0$ in the Boltzmann-Grad scaling, we have

$$\frac{(N-s)!}{(N-s-n)!}\varepsilon^{n(d-1)} \to 1\,.$$

Defining

$$\begin{aligned}
\overline{J}^{R,\delta}_{s,n}(t,J,M)(X_s) = &\int_{B_R\setminus\mathcal{M}_s(X_s)} dV_s\varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)} dT_n\\
&\int_{{}^c\mathcal{B}_s(Z^0_s(t_1))} d\nu_{s+1}dv_{s+1}\left(\nu_{s+1}\cdot(v_{s+1}-v_{m_1}(t_1))\right)_{j_1}\\
&\quad\dots\int_{{}^c\mathcal{B}_{s+n-1}(Z^0_{s+n-1}(t_n))} d\nu_{s+n}dv_{s+n}\left(\nu_{s+n}\cdot(v_{s+n}-v_{m_n}(t_n))\right)_{j_n}\\
&\quad\times \mathbb{1}_{E_0(Z_{s+n}(0))\leq R^2}\mathbb{1}_{Z^\varepsilon_{s+n}(0)\in\mathcal{G}_{s+n}(\varepsilon_0/2)}f_0^{(s+n)}(Z_{s+n}(0))\,,
\end{aligned}\tag{10.2.9}$$

we have the following obvious convergence.

Lemma 10.2.2. — *In the Boltzmann-Grad scaling $N\varepsilon^{d-1}=1$,*

$$|\mathbb{1}_{\Delta^X_s}(\widetilde{J}^{R,\delta}_{s,n}-\overline{J}^{R,\delta}_{s,n})(t,J,M)(X_s)| \leq C\frac{s(s+n)}{N}\|\varphi\|_{L^\infty(\mathbf{R}^{ds})}|\!|\!|F_{N,0}|\!|\!|_{0,\beta_0,\mu_0}\,.$$

10.2.3. Conclusion. — We can now compare the definition (9.2.13) of $J^{0,R,\delta}_{s,n}(t,J,M)$

$$\begin{aligned}
J^{0,R,\delta}_{s,n}(t,J,M)(X_s) = &\int_{B_R\setminus\mathcal{M}_s(X_s)} dV_s\varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)} dT_n\\
&\int_{{}^c\mathcal{B}_s(Z^0_s(t_1))} d\nu_{s+1}dv_{s+1}((v_{s+1}-v^0_{m_1}(t_1)\cdot\nu_{s+1})_{j_1}\\
&\quad\dots\int_{{}^c\mathcal{B}_{s+n-1}i(Z^0_{s+n-1}(t_n))} d\nu_{s+n}dv_{s+n}((v_{s+n}-v^0_{m_n}(t_n)\cdot\nu_{s+n})_{j_n}\\
&\mathbb{1}_{E_0(Z^0_{s+n}(0))\leq R^2}f_0^{(s+n)}(Z^0_{s+n}(0))\,.
\end{aligned}$$

and the formulation (10.2.9) for the approximate BBGKY hierarchy

$$\begin{aligned}
\overline{J}^{R,\delta}_{s,n}(t,J,M)(X_s) = &\int_{B_R\setminus\mathcal{M}_s(X_s)} dV_s\varphi_s(V_s)\int_{\mathcal{T}_{n,\delta}(t)} dT_n\\
&\int_{{}^c\mathcal{B}_s(Z^0_s(t_1))} d\nu_{s+1}dv_{s+1}\left(\nu_{s+1}\cdot(v_{s+1}-v_{m_1}(t_1))\right)_{j_1}\\
&\quad\dots\int_{{}^c\mathcal{B}_{s+n-1}(Z^0_{s+n-1}(t_n))} d\nu_{s+n}dv_{s+n}\left(\nu_{s+n}\cdot(v_{s+n}-v_{m_n}(t_n))\right)_{j_n}\\
&\quad\times \mathbb{1}_{E_0(Z_{s+n}(0))\leq R^2}\mathbb{1}_{Z^\varepsilon_{s+n}(0)\in\mathcal{G}_{s+n}(\varepsilon_0/2)}f_0^{(s+n)}(Z_{s+n}(0))\,.
\end{aligned}$$

Lemma 10.1.1 implies that at time 0 we have

$$|X_{s+n}(0)-X^0_{s+n}(0)| \leq C(R,\eta)n\varepsilon\,,\quad\text{and}\quad V_{s+n}(0)=V^0_{s+n}(0)\,.$$

Provided that $f_0^{(s+n)}$ is continuous, we then obtain the expected convergence and this concludes the proof of Theorem 4.

Notice for instance that if $f_0^{(s+n)}$ is Lipschitz, then we have the following estimate.

Proposition 10.2.1. — *In the Boltzmann-Grad scaling* $N\varepsilon^{d-1} = 1$,

$$|1\!\!1_{\Delta_s^X(\varepsilon_0)}(I^0 - \overline{J})(t, X_s)| \leq C(R,\eta) n\varepsilon \|\nabla_{X_{s+n}} f_0^{(s+n)}\|_{L^\infty} ,$$

uniformly in $t \in [0,T]$ *and* $X_s \in \mathbf{R}^{ds}$.

CHAPTER 11

CONCLUDING REMARKS

11.1. On the convergence rate : the particular case of hard spheres

The method of proof described in this text gives actually a more precise statement regarding the convergence then Theorem 4 : gathering all the estimates together, we indeed obtain a rate of convergence. For general short-range potentials of interaction, this rate is not completely explicit since the constant arising in Proposition 9.1.1 in the estimate of pathological sets depends on Φ, η and R through the cross-section b.

This constant can be made explicit in particular cases, especially in the simple case of hard spheres since the deflection angle ω and the normal ν coincide.

Note that all the arguments can be reproduced in this case once the dynamics for fixed N and ε is well-defined (without multiple collisions - see Chapter 2). Moreover there are important simplifications :

- True marginals coincide with truncated marginals because of the non penetration condition. In particular, there is no more need of cluster expansions, which simplifies a little bit the existence proof, and Proposition 8.1.1 is no more relevant.
- The scattering operator is completely explicit since $\omega = \nu$ in formulas (3.2.2). In particular the cross-section
$$b(v_1 - v_2, \omega) = ((v_1 - v_2) \cdot \omega)_+ \,.$$
As mentioned above, this enables us to get a constant $C(R, \eta, \Phi) = C_d$ in Lemma 9.1.4 that depends only on the dimension d. Since we use Lemma 9.1.4 with $\rho = 12Ra/\varepsilon_0 + 12R\varepsilon_0/\delta$ in the proof of Proposition 9.1.1, this gives an error
$$C_0 R^d \left(\eta^{d-1} + \left(\frac{a}{\varepsilon_0} + \frac{\varepsilon_0}{\delta} \right)^{d-1} \right) \text{ with } \eta\delta >> \varepsilon_0 \,.$$
Note that we can choose $\eta = C\frac{\varepsilon_0}{\delta}$ for C sufficiently large.
- Collisions are pointwise and instantaneous $t_\varepsilon \equiv 0$, which makes the proof of Lemma 10.1.1 on the divergence of trajectories very easy. Indeed, the distance between the BBGKY and Boltzmann pseudo-trajectories increases at most of ε at each collision!

Let us then gather all the estimates together. We assume for the sake of simplicity that we start from an almost factorized initial data, i.e. a BBGKY initial data obtained from a tensor product by the conditioning process described in Chapter 7.

From the arguments of Proposition 8.2.1, 8.3.1 and 8.4.1, we find a first error term

$$e_1 \leq C_0 \left(\left(\frac{2}{3}\right)^n e^{s\mu(t)} + e^{-C'R^2} + n\delta \right) ,$$

where C_0 depends only on the L^∞ norm of f_0. Then, from Proposition 9.1.1, we obtain as discussed above the error term

$$e_2 \leq C_0 R^d \left(\frac{a}{\varepsilon_0} + \frac{\varepsilon_0}{\delta} \right)^{d-1} .$$

Finally, we have to take into account the error coming from the initial data, estimated in Chapter 7 and Lemma 10.2.1:

$$e_3 \leq C_0 \frac{R^{dn} T^n}{n!} (s+n)\varepsilon \, ,$$

the error coming from the prefactors of the collision operators

$$e_4 \leq C_0 s(s+n)\varepsilon^2$$

and the error coming from the divergence of trajectories, which can be estimated if f_0 is Lipschitz as follows

$$e_5 \leq C \|f_0\|_{W^{1,\infty}} \frac{R^{dn} T^n}{n!} (s+n)\varepsilon \, .$$

Therefore, choosing

$$n \sim C_1 |\log \varepsilon|, \quad R^2 \sim C_2 |\log \varepsilon|$$

for some sufficiently large constants C_1 and C_2, and

$$\delta = \varepsilon^{(d-1)/(d+1)}, \quad \varepsilon_0 = \varepsilon^{d/(d+1)}$$

we find that the total error is smaller than $C\varepsilon^\alpha$ for any $\alpha < (d-1)/(d+1)$.

11.2. On the time of validity of Theorems 2 and 4

Let us first note that, for any fixed N, the BBGKY hierarchy has a global solution since it is equivalent to the Liouville equation in the phase space of dimension $2Nd$, which is nothing else than a linear transport equation. The fact that we obtain a finite life span is therefore due to the functional spaces $\mathbf{X}_{\varepsilon,\beta,\mu}$ we consider. Belonging to such a functional space requires indeed a strong control on the high order correlations. The estimates we have written show actually (see Corollary 6.1.2) that the time of validity of Theorem 2 depends on μ_0, which measures the logarithmic growth of the initial marginals (that is the size of f_0 for factorized initial data).

An important point is that the time of convergence is exactly the time of existence. By definition of the functional spaces, we are indeed in a situation where the high order correlations can be neglected (see Proposition 7.2.1), so that we only have to study the dynamics of a finite system of particles. The term-by-term convergence relies then on geometrical properties of the transport in the whole space, which do not introduce any restriction on the time of convergence.

A natural question is therefore to know whether or not it is possible to increase the time of existence and thus the time of convergence. The interpretation we have of the iterated Duhamel formula in terms of pseudo-trajectories, for which at each time step a *new* particle (independent from all the others) is added and interacts with one of the previous ones, shows that one cannot hope to prove by this method a convergence theorem for larger times than a fraction of the time at which each particle has undergone one collision.

The first trivial remark is that after $N-1$ iterations, independence does not hold any longer. Indeed, representing each collision by a strap between the two involved particles, we see that
– either there is a chain connecting all the particles,
– or there are closed subchains, corresponding to pathological dynamics involving recollisions.
It is actually known (see for instance [**6**]) that after cN time steps with $1/2 < c < 1$, there is a change of phase and the appearance of a "giant component" in the set of N particles, meaning a set of $\alpha(c)N$ particles, with $\alpha(c) \to 1$ as $c \to 1$ which have interacted either directly, or indirectly through other particles, so independence does not hold anymore, after $N/2$ time steps.

In other words this means that, starting from such a system of N particles, we can expect to increase the time of convergence (and thus the time of existence) only if we can prove that, after a short time, particles go at infinity in different directions, and thus do not encounter each other any more, i.e. the dynamics reduces to free transport.

If we want to establish the validity of the Boltzmann equation for longer times, we have therefore to start with systems of particles which contain much more particles, but with the same average density (in order that the collision cross-section remains bounded). The difficulties are then to prove that
– the density of particles remains locally bounded, so that the asymptotics is still governed by the Boltzmann-Grad scaling;
– the spatial dispersion creates some mixing mechanism, which implies that particles entering a collision are always independent.
Note that a simple way to get rid of the first problem is to consider periodic distributions of particles. In that case, the challenge is to understand how the dispersion associated to free transport could help for the propagation of chaos, which implies more or less to study the spatial decay of correlations.

11.3. More general potentials

A first natural extension to this work concerns the case of a compactly supported, repulsive potential, but no longer satisfying (1.2.1) of Assumption 1.2.1. As explained in Chapter 3, that assumption guarantees that the cross section is well defined everywhere, since the deflection angle is a one-to-one function of the impact parameter. If that is no longer satisfied, then one expects that additional decompositions are necessary, and resummation procedures need to be justified (see [**36**]).

From a physical point of view it would be more interesting to study the case of long-range potentials. Then the cross section actually becomes singular, so a different notion of limit must be considered, possibly in the spirit of Alexandre and Villani [**3**]. One intermediate step, as in [**15**], would be to extend this work to the case when the support of the potential goes to infinity with the number of particles. Then one could try truncating the long-range potential and showing that the tail of the potential has very little effect in the convergence.

Note that in the case when grazing collisions become predominant, then the Boltzmann equation should be replaced by the Landau equation, whose derivation is essentially open; a first result in that direction was obtained very recently by A. Bobylev, M. Pulvirenti and C. Saffirio in [**4**], where a time zero convergence result is established.

11.4. Other boundary conditions

As it stands, our analysis is restricted to the whole space (namely $X_N \in \mathbf{R}^{dN}$). It is indeed important that free flow corresponds to straight lines (see in particular Lemmas 9.1.3 and 9.1.4 as well more generally as the analysis of pathological trajectories in Chapter 9).

It would be very interesting to generalize this work to more general geometries. A first step in that direction would be to study the case of periodic flows in X_N. The geometric lemmas must be adapted to that framework, and in particular it appears that a finite life span must a priori be given before the surgery of the collision integrals may be performed. The case of a general domain is again much more complicated, and results from the theory of billiards would probably need to be used.

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NOTATION INDEX